II. *Researches in Vortex Motion.*—Part III. *On Spiral or Gyrostatic Vortex Aggregates.*[*]

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(Plates 1, 2.)

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[*] Parts I. and II., 'Phil. Trans.,' 1884 (I.), 1885 (II.)

THE chief part of the following investigation (Sects. i. and iii.) was undertaken with the view of discovering whether it was possible to imagine a kind of vortex motion which would impress a gyrostatic quality which the forms of vortex aggregates hitherto known do not possess. The other part (Sect. ii.) deals with the non-gyrostatic vortex aggregates, the discovery of which we owe to HILL,* and investigates the conditions under which two or more aggregates may be combined into one. It is shown that it is allowable to suppose one or more concentric shells of vortex aggregates to be applied over a central spherical nucleus, subject to one relation between the radii and the vorticities. In all cases the vorticities must be in opposite directions in alternate shells. The special case when the aggregates are built up of the same vortical matter is considered, and the magnitudes of the radii and the positions of the equatorial axes determined. The cases of motion in a rigid spheroidal shell and of dyad spheroidal aggregates are also considered.

The chief part of the paper refers to gyrostatic aggregates. The investigation has brought to light an entirely new system of spiral vortices. The general conditions for the existence of such systems, when the motion is symmetrical about an axis, are determined in Sect. i., and are worked out in more detail for a particular case of spherical aggregate in Sect. iii. It is found that the motion in meridian planes is determined from a certain function ψ in the usual manner. The velocity along a parallel of latitude is given by $v = f(\psi)/\rho$ where ρ is the distance of the point from the axis. The function ψ, however, does not depend on the differential equation of the ordinary non-spiral type, but is a solution of the equation

$$\frac{d^2\psi}{dr^2} + \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} - \frac{\cot\theta}{r^2}\frac{d\psi}{d\theta} = \rho^2 \mathrm{F} - f\frac{df}{d\psi},$$

where F and f are both functions of ψ. The case F and $f df/d\psi$ both uniform is briefly treated. It refers to a spiral aggregate with a central solid nucleus, and is not of great interest. The case F uniform and $f \propto \psi$ is treated more fully. If $f = \lambda\psi/a$ where a is the radius of the aggregate

$$\psi = \mathrm{A}\left\{\mathrm{J}_2\left(\frac{\lambda r}{a}\right) - \frac{r^2}{a^2}\mathrm{J}_2\lambda\right\}\sin^2\theta.$$

The most striking and remarkable fact brought out is that with increasing parameter λ, we get a periodic system of families of aggregates. The members of each family differ from one another in the number of layers and equatorial axes they possess. I have ventured to call them singlets, doublets, triplets, &c., in contradistinction to the more or less fortuitous and arbitrary compounds dealt with later, and which I have named monads, dyads, triads, &c. Of these families two are investigated more in detail than the others. In one family (the λ_2 family) all the members remain at rest in the surrounding fluid. In the other (the λ_1 family) the distinguishing feature common to all the members is that the stream lines and the vortex lines are coincident.

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* "On a Spherical Vortex," 'Phil. Trans.,' A, vol. 185, 1894.

The parameter λ defines the total angular pitch of the stream lines, on the outer current-sheet, viz., up the polar axis and down the outside; although in the aggregates with more than one axis these lines are not one continuous stream line. The first aggregates—with $\lambda < 5{\cdot}7637$ (the first λ_2 parameter)—behave abnormally. Beyond these we get successive series, in one set of which the velocity of translation is in the same direction as the polar motion of the central nucleus, in the alternate set the velocity is opposite, and the aggregate regredes in the fluid as compared with its central aggregate (see fig. 3, Plate 1). The physical analogue of these aggregates is obvious. It is specially enlarged upon in the abstract.*

Suppose we set ourselves the problem of making a set of aggregates with greater and greater angular pitch. As we do so we shall find that as the pitch increases the equatorial axis contracts, and the surface velocity diminishes. On the outer layers (ring shaped) the spiral is chiefly produced on the inner side facing the polar axis, until on the boundary itself the stream lines flow in meridians, and the twist is altogether on the polar axis. The pitch can be increased up to a certain degree. As this is done, the stream lines and vortex lines fold up towards one another, coincide at a certain pitch, and exchange sides. When an external angular pitch of about 330° is attained it is impossible to go further if a simple aggregate is desired. If a higher pitch is desired it is attained by taking it in two parts. First, a central spherical nucleus of the same nature as the former, in which a portion of the twist is produced, and outside this a spherical shell, in which the spirals have the same direction of twist, and complete the pitch to the desired amount but in which the spirals are traversed in the opposite direction. With increasing pitch this layer becomes thicker, and its equatorial axis contracts relatively to the mid-point of the shell until another limit is reached; the stream and vortex lines again fold together, cross, and expand as this second limit is reached. If a larger pitch still is desired there must be a third layer, and so on. The first coincidence of vortex and stream lines takes place for an aggregate whose pitch is $257^\circ{\cdot}27'$. Whenever a maximum pitch is attained the aggregate is at rest in the fluid. This is first attained for an external pitch of $330^\circ{\cdot}14'$. Beyond this there are two equatorial axes. For an external pitch of $442^\circ{\cdot}37'$ the stream and vortex lines again coincide, the internal nucleus gives $257^\circ{\cdot}27'$ of the pitch and the outer shell the remainder, and so on.

At the end a theory of compound aggregates is developed similar to that in Sect. ii. for non-gyrostatic vortices. It is not worked out in detail in the present communication, but the conditions are determined for dyad compounds, whilst a similar theory holds for triad and higher ones. Each element of a poly-ad may consist of singlets, doublets, &c. The equations of condition leave three quantities arbitrary—

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* 'Roy. Soc. Proc.,' vol. 62, p. 332.

as, for instance, ratio of volumes, ratio of primary cyclic constants, ratio of secondary cyclic constants. The full development of this theory is, however, left for a future communication. It is clear that spiral or gyrostatic vortex aggregates are not confined to forms symmetrical about an axis. Their theory is however much more complicated.

If we take any particular spherical aggregate with given λ and primary cyclic constant (μ), the energy is determinate. We may, however, alter the energy. If it be increased, the spherical form begins to open out into a ring form, whose shape and properties have not yet been investigated. If the energy be increased sufficiently the aperture becomes large compared with the thickness of the rotational core, and approximate calculation can be applied. The differential equation for ψ is given in Sect. i., but its development is left for a future occasion. After that I hope to deal with the question of stability, and then more fully with that of the conditions of combination. The new field opens up so many questions of interest that other workers in it are welcomed.

Section i.—*General Theorems.*

1. To give an idea of the nature of the motions considered in the present investigation, consider the case of motion of an infinitely long cylindrical vortex of sectional radius a. The velocity perpendicular to the axis inside the vortex will be of the form $v = f(r)$, where $f(0) = 0$. Outside it will be given by $v = \mathrm{V}a/r$, where $\mathrm{V} = f(a)$.

We may, however, have a motion in which the fluid moves parallel to the axis inside the cylinder with rest outside. The velocity will be of the form $u = \mathrm{F}(r)$ inside, where $\mathrm{F}(a) = 0$, and zero outside. Both $f(r)$ and $\mathrm{F}(r)$ are arbitrary functions subject only to the conditions $f(0) = 0$ and $\mathrm{F}(a) = 0$.

Putting aside for the present the question of the stability of these simple motions or of their resultant, it is clear that if we superpose the two we get another state of motion in which we have vortex-filaments in the shape of helices lying on concentric cylindric surfaces. The problem to be considered is whether it is possible to conceive a similar superposition of two motions in the case of any vortex aggregate whose motions are symmetric about an axis.

There are an infinite number of either ring-shaped vortices, or singly connected aggregates (of which HILL'S vortex may serve as a type), differing from one another in the law of vorticity of the different parts—the most important being those in which the vorticity is uniform. The motions in all these are known in terms of the stream function ψ. The value of ψ is however at present only actually known for an infinitely thin ring-filament or for a spherical aggregate.

2. We are to consider two superposed motions. The one component is in meridian planes through an axis and can be defined in terms of the stream-function ψ.* The

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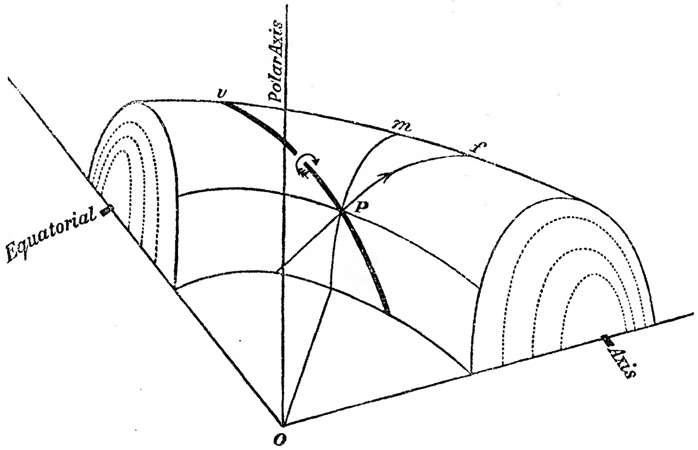
* Throughout ψ is taken as the total flow *up* through the circle whose radius is ρ. In other words the velocity perpendicular to ds is $\dfrac{1}{2\pi\rho}\dfrac{d\psi}{ds}$.

other component is everywhere perpendicular to these meridian planes. The vortex aggregates will be moving with rectilinear translation through the fluid with a velocity calculable, when the distribution of vortex motion is known, by HELMHOLTZ'S method. Bring the aggregate to rest by impressing everywhere a velocity equal and opposite to the velocity of translation. The motion then consists of a flow up through the centre in the direction of previous translation, the fluid then streaming (in this most general case) in spirals round a certain circle. The circle may conveniently be called the *equatorial axis* of the aggregate. The line of symmetry through the centre in the direction of translation may then be termed the *polar axis*. Whether we deal with ring-shaped or singly connected aggregates, the surfaces ψ will always be ring-shaped inside. In fact they are so also at the boundary, for the surface value of ψ really consists in the latter case of the outer boundary together with the polar axis.

3. Conceive now the aggregate divided up into a large number of ring-surfaces given by values of a parameter ψ differing by $d\psi$, and confine attention to what is going on between the two surfaces ψ and $\psi + d\psi$. We shall suppose ψ to increase as we pass from the outside inwards. Let dn denote the distance at a point between the surfaces ψ and $\psi + d\psi$, dn to be measured also inwards. In the shell considered the lines of flow will be spiral, and the vortex-filaments also spirals, as indicated in the figure, the thin line Pf representing a line of flow, the thick Pv a vortex-filament,

Fig. 1.



and the line Pm a meridian section. Denote the velocity at P by v and the angle it makes with the meridian by ϕ. Also let ω denote the molecular rotation at P, and χ the angle the filament makes with the meridian—estimated positive when on the opposite side of the meridian to v.

Consider the flow between the two surfaces ψ and $\psi + d\psi$ across the "parallel of latitude" through P. The total flow must be the same for every parallel. The area through which the flow takes place is $2\pi\rho dn$, where ρ is the distance of P from the polar axis. Hence $2\pi\rho v \cos\phi dn$ is constant over the surface ψ. It must therefore be of the form $f(\psi)\,d\psi$. So far ψ is only defined as the parameter which determines the particular surface. Choose the parameter so that $f(\psi) = 1$. ψ is then analogous to the stream-function in the simple case. It acts in fact as the stream-function for the component of velocity $v \cos\phi$. Similar reasoning leads to the conclusion that $\omega\rho \cos\chi dn$ is also of the form $f(\psi)\,d\psi$, say $f_1\,d\psi$. Hence

$$2\pi\rho v \cos\phi dn = d\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$2\pi\rho\omega \cos\chi dn = f_1\,d\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

We started with the supposition that the stream-lines and vortex-lines must lie on the same surfaces ψ. In other words, there must be no component rotation perpendicular to ψ. This may be expressed in other words by the statement that the circulation round any circuit drawn wholly on ψ must vanish. Take for this circuit any two parallels of latitude. The condition gives that the flow along one must equal the flow along the other. In other words, the flow round a parallel of latitude must be the same for all parallels on the same surface ψ. Hence

$$2\pi\rho v \sin\phi = f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where f is a function of ψ.

Equations 1, 2, 3 give conditions which any motion possible between any two given surfaces ψ and $\psi + d\psi$ must satisfy. In our case, however, the motions in the separate shells must fit together. We may regard the vortex-filaments as due to the velocities in two successive shells, *or* as due to the different velocities on the inner and outer surfaces of the same shell—the velocities on the inner surface of one being the same as on the outer of the next succeeding shell. If now ω_1 be any component of a filament, and $d\mathrm{A}$ the area perpendicular to ω_1, the value of $\omega_1\,d\mathrm{A}$ is given by half the circulation round $d\mathrm{A}$. Apply this to the two components $\omega \cos\chi$ along a meridian and $\omega \sin\chi$ along a parallel of latitude. As a circuit for $\omega \cos\chi$ take two parallels one on ψ and the other on $\psi + d\psi$. The flow along the first is $2\pi\rho v \sin\phi$ and along the latter

$$2\pi\rho v \sin\phi + 2\pi \frac{d}{dn}(\rho v \sin\phi)\,dn.$$

Hence

$$2\omega \cos\chi \,.\, 2\pi\rho dn = -2\pi \frac{d}{dn}(v\rho \sin\phi)\,dn.$$

But by (3),

$$2\pi\rho v \sin\phi = f.$$

Hence

$$4\pi\rho\omega \cos\chi = -\frac{df}{dn} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Comparing with (2) it follows that

$$f_1 \frac{d\psi}{dn} = -\tfrac{1}{2}\frac{df}{dn},$$

or

$$f_1 = -\tfrac{1}{2}\frac{df}{d\psi}.$$

We may regard then Eq. (2) as replaced by (4), which includes it as the greater does the less.

For the circuit for $\omega \sin\chi$ take a small circuit formed by a small arc ds of a meridian PP′ on ψ, the normals (dn) at P, P′ and the portion of the meridian arc on $\psi + d\psi$ cut off by these normals. The flow along the normals dn is zero. Along ds it is $v\cos\phi\, ds$; along ds' it is

$$v\cos\phi\, ds + \frac{d}{dn}(v\cos\phi\, ds)\, dn.$$

The area of the cross-section of $\omega\sin\chi$ is $dn\, ds$.
Hence

$$2\omega\sin\chi\, dn\, ds = -\frac{d}{dn}(v\cos\phi\, ds)\, dn.$$

But by (1),

$$v\cos\phi = \frac{1}{2\pi\rho}\frac{d\psi}{dn};$$

therefore

$$4\pi\omega\sin\chi\, ds = -\frac{d}{dn}\left(\frac{1}{\rho}\frac{d\psi}{dn}\, ds\right).$$

Since $d\psi/ds = 0$ ψ will give any component of velocity *in the meridian plane* in the same way as the ordinary stream-function.

4. It will often be found advantageous to express ψ in terms of curvilinear co-ordinates. Denote these by u, v. Displacements perpendicular to the u will be denoted by dn, and to v by dn', to be estimated positive in the directions in which u, v respectively increase.

The differential equation satisfied by ψ is found by expressing the circulation round a small area bounded by the curves u, $u + du$, v, $v + dv$. Let ω_1 ($= \omega\sin\chi$) denote the rotation at a point of the area. We shall regard this as positive when it goes clockwise. The circulation is then $2\omega_1 \times$ area $= 2\omega_1\, dn, dn'$.

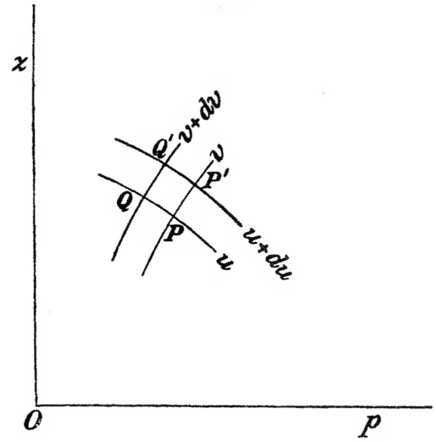
The velocities along PQ, PP′ (see fig. 2) are respectively

$$\frac{1}{2\pi\rho}\frac{d\psi}{dn}, \qquad -\frac{1}{2\pi\rho}\frac{d\psi}{dn'},$$

The flows along them are therefore (clockwise)

$$\frac{1}{2\pi\rho}\frac{d\psi}{dn}\, dn' \text{ and } +\frac{1}{2\pi\rho}\frac{d\psi}{dn'}\, dn.$$

Fig. 2.



Hence the total flow round PQQ′P′ is

$$-\frac{d}{du}\left(\frac{1}{2\pi\rho}\frac{d\psi}{dn}\,dn'\right)du-\frac{d}{dv}\left(\frac{1}{2\pi\rho}\frac{d\psi}{dn'}\,dn\right)dv,$$

or

$$-\frac{d}{du}\left(\frac{1}{2\pi\rho}\frac{d\psi}{du}\frac{du}{dn}.\frac{dn'}{dv}\right)du\,dv-\frac{d}{dv}\left(\frac{1}{2\pi\rho}\frac{d\psi}{dv}.\frac{dv}{dn'}.\frac{dn}{du}\right)du\,dv.$$

But this is $2\omega_1\,dn\,dn^1$. Hence

$$\frac{d}{du}\left(\frac{1}{\rho}\frac{d\psi}{du}.\frac{du}{dn}.\frac{dn'}{dv}\right)+\frac{d}{dv}\left(\frac{1}{\rho}\frac{d\psi}{dv}\frac{dv}{dn'}.\frac{dn}{du}\right)=-4\pi\omega_1\frac{dn}{du}\frac{dn'}{dv}$$
$$=-4\pi\omega\sin\chi\frac{dn}{du}\frac{dn'}{dv}\quad.\quad(5).$$

In many cases $\rho+z\iota=f(u+v\iota)$, giving $du/dn=dv/dn'$, and the equation simplifies to

$$\frac{d}{du}\left(\frac{1}{\rho}\frac{d\psi}{du}\right)+\frac{d}{dv}\left(\frac{1}{\rho}\frac{d\psi}{dv}\right)=-4\pi\omega_1\left(\frac{dn}{du}\right)^2.$$

The following cases will be required :—

(1) *Cylindrical co-ordinates.* (ρ, z),

$$du=d\rho=dn\qquad dv=dz=dn',$$

and

$$\frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d\psi}{d\rho}\right)+\frac{1}{\rho}\frac{d^2\psi}{dz^2}=-4\pi\omega\sin\chi,$$

or

$$\frac{d^2\psi}{d\rho^2}-\frac{1}{\rho}\frac{d\psi}{d\rho}+\frac{d^2\psi}{dz^2}=-4\pi\rho\omega\sin\chi\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad(6).$$

(2) *Polar Co-ordinates.* (r, θ),

$$\rho = r \sin\theta \qquad du = dr = dn \qquad dv = d\theta \qquad dn' = r\,d\theta,$$

and

$$\frac{d}{dr}\left(\frac{r}{\rho}\frac{d\psi}{dr}\right) + \frac{d}{d\theta}\left(\frac{1}{r\rho}\frac{d\psi}{d\theta}\right) = -\,4\pi r\omega \sin\chi,$$

or

$$\frac{d^2\psi}{dr^2} + \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} - \frac{\cot\theta}{r^2}\frac{d\psi}{d\theta} = -\,4\pi\rho\omega\sin\chi \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

(3) *Spheroids.*

(α) Prolate. Here $\rho + z\iota = \lambda \sinh(u + v\iota)$,
whence

$$\rho = \lambda \sinh u \cos v, \qquad z = \lambda \cosh u \sin v.$$

The surfaces u, v are respectively the ellipses and hyperbolas

$$\frac{\rho^2}{\sinh^2 u} + \frac{z^2}{\cosh^2 u} = \lambda^2 \quad \text{and} \quad \frac{z^2}{\sin^2 v} - \frac{\rho^2}{\cos^2 v} = \lambda^2,$$

u increases from 0 at the origin to ∞ at an infinite distance; v increases from $-\frac{1}{2}\pi$ at points on the negative part of the axis of z, through 0 for points on the equatorial plane to $\frac{1}{2}\pi$ at points on the positive part of the axis of z.

Again

$$\left(\frac{dn}{du}\right)^2 = \left(\frac{d\rho}{du}\right)^2 + \left(\frac{dz}{du}\right)^2 = \frac{d}{du}(\rho + z\iota)\frac{d}{du}(\rho - z\iota)$$
$$= \lambda^2 \cosh(u + v\iota)\cosh(u - v\iota)$$
$$= \lambda^2(\cosh^2 u - \sin^2 v).$$

Hence the differential equation is (writing C and S for cosh u, sinh u),

$$\frac{1}{\cos v}\frac{d}{du}\left(\frac{1}{\mathrm{S}}\frac{d\psi}{du}\right) + \frac{1}{\mathrm{S}}\frac{d}{dv}\left(\frac{1}{\cos v}\frac{d\psi}{dv}\right) = -\,4\pi\lambda^3\omega\sin\chi\,(\mathrm{C}^2 - \sin^2 v) \quad . \quad . \quad (8).$$

(β) Oblate. Here $\rho + z\iota = \lambda\cosh(u + v\iota)$

$$\rho = \lambda\cosh u \cos v, \qquad z = \lambda \sinh u \sin v,$$
$$\left(\frac{dn}{du}\right)^2 = \lambda^2 \sinh(u + v\iota)\sinh(u - v\iota)$$
$$= \lambda^2(\cosh^2 u - \cos^2 v),$$

and the differential equation is

$$\frac{1}{\cos v}\frac{d}{du}\left(\frac{1}{\mathrm{C}}\frac{d\psi}{du}\right) + \frac{1}{\mathrm{C}}\frac{d}{dv}\left(\frac{1}{\cos v}\frac{d\psi}{dv}\right) = -\,4\pi\lambda^3\omega\sin\chi\,(\mathrm{C}^2 - \cos^2 v) \quad . \quad . \quad (9).$$

(4) *Toroidal Functions.*—Here [‘Phil. Trans.,’ 1881, Part III., p. 614]

$$u + v\iota = \log \frac{\rho + a + z\iota}{\rho + a + z\iota}, \qquad \rho = a\frac{\sinh u}{\cosh u - \cos v}, \qquad \frac{du}{dn} = \frac{\sinh u}{\rho},$$

whence

$$\frac{d}{dv}\left(\frac{\mathrm{C} - \cos v}{\mathrm{S}}\frac{d\psi}{du}\right) + \frac{1}{\mathrm{S}}\frac{d}{dv}\left((\mathrm{C} - \cos v)\frac{d\psi}{dv}\right) = -4\pi a\frac{\rho^2}{\mathrm{S}^2}\omega \sin \chi,$$

$$= -\frac{4\pi a^3}{(\mathrm{C} - \cos v)^2}\omega \sin \chi \quad . \quad (10).$$

5. Equations 1, 3, 4, 5 or 6 give the conditions for a possible motion. It is open to us to choose ψ arbitrarily. In this case the equations give v, ω, χ, ϕ. The motion is instantaneously possible, but in general it will at once proceed to change the configuration—the motion will not be steady. The application of this theory to values of ψ which are already known (HILL's vortex for example) leads to interesting results, but the absence of steadiness robs the theory of importance. If we impose the condition of steady motion, it is no longer open to us to choose ψ at will. Let us then impose this condition. The condition that the motion shall be steady involves:—

(1) ψ must be a surface containing both vortex-lines and stream-lines.

This is already the case.

(2) $v\omega \sin(\phi + \chi)\, dn$ must be constant over the surface.

It must therefore be of the form $\mathrm{F}d\psi$, where F is a function of ψ. Hence

$$v\omega \sin(\phi + \chi) = \mathrm{F}\frac{d\psi}{dn} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Expanding this, and substituting from 1, 3, 4, 7,

$$-f\frac{df}{d\psi} - \left\{\frac{d^2\psi}{dr^2} + \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} - \frac{\cot\theta}{r^2}\frac{d\psi}{d\theta}\right\} = 8\pi^2\rho^2\mathrm{F},$$

or

$$\frac{d^2\psi}{dr^2} + \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} - \frac{\cot\theta}{r^2}\frac{d\psi}{d\theta} = -8\pi^2\rho^2\mathrm{F} - f\frac{df}{d\psi} \quad . \quad . \quad . \quad . \quad (12),$$

where f and F are arbitrary functions of ψ. Choosing these, equation 12 will give the type of ψ.*

We proceed to apply these general theorems to certain special cases of spherical aggregates. In order to exemplify the method employed we will take first the case in which there is no secondary spin, the type in which HILL's spherical vortex is the simplest case.

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* For another proof of this equation, due to one of the referees, see end of present paper.

Section ii.—*Aggregates with no Secondary Spin* ($f = 0$) *and with Uniform Vorticity.*

6. We begin with the spherical aggregate, the simplest type of which is the HILL's vortex. The equation for ψ is that given by equation 7, in which ω is put $k\rho$ where k is uniform and $\chi = \frac{1}{2}\pi$. It is

$$\frac{d^2\psi}{dr^2} + \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} - \frac{\cot\theta}{r^2}\frac{d\psi}{d\theta} = -4\pi k\rho^2 = -4\pi kr^2\sin^2\theta,$$

in which θ is measured from the pole to the equator. A particular solution of this is $-\frac{1}{2}\pi kr^4\sin^4\theta$.

In

$$\frac{d^2\psi}{dr^2} + \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} - \frac{\cot\theta}{r^2}\frac{d\psi}{d\theta} = 0,$$

put $\psi = r^n Z_n$, Z_n being a function of θ only. Then

$$\frac{d^2Z_n}{d\theta^2} - \cot\theta\frac{dZ_n}{d\theta} + n(n-1)Z_n = 0.$$

The integral of this is

$$Z_n = -\sin\theta\frac{dP_{n-1}}{d\theta},$$

where P_{n-1} is a zonal harmonic of degree $n - 1$.

Hence the general solution of the equation in ψ is

$$\psi = -\tfrac{1}{2}\pi kr^4\sin^4\theta + \Sigma\left(A_n r^n + \frac{B}{r^{n-1}}\right)Z_n.$$

Since

$$P_n = \frac{1.3.5\ldots(2n-1)}{n\,!}\left\{\cos^n\theta - \frac{n(n-1)}{2(2n-1)}\cos^{n-2}\theta + \ldots\right\}$$

the values of Z_n are easily found, except for Z_1 or Z_0. It is easily found from the direct equation in this case that $Z_1 = Z_0 = \cos\theta$. The following results are easily deduced :—

$$Z_2 = \sin^2\theta, \qquad Z_3 = 3\sin^2\theta\cos\theta,$$

$$Z_4 = \tfrac{3}{2}(4\sin^2\theta - 5\sin^4\theta), \qquad \sin^4\theta = \tfrac{4}{5}Z_2 - \tfrac{2}{15}Z_4.$$

Consider now first the case of a homogeneous spherical aggregate. In this case the functions $\frac{B}{r^{n-1}}Z_n$ apply only to the space outside, and Ar^nZ_n to the space inside. Let ψ_1 denote the value of ψ inside and ψ_2 outside. Hence

$$\psi_1 = -\tfrac{1}{2}\pi k r^4 \sin^4\theta + \Sigma A_n r^n Z_n$$

$$\psi_2 = \Sigma \frac{B_n}{r^{n-1}} Z_n.$$

Let a denote the radius of the sphere. Along the boundary of the sphere $\psi_1 = \psi_2$, and also $d\psi_1/dr = d\psi_2/dr$. Expressing $\sin^4\theta$ in terms of Z_2 and Z_4,

$$\psi_1 = -\tfrac{2}{5}\pi k r^4 Z_2 + \tfrac{1}{15}\pi k r^4 Z_4 + \Sigma A_n r^n Z_n.$$

The term in $\frac{1}{15}\pi k r^4 Z_4$ may be supposed merged in $A_4 r^4 Z_4$, and may therefore be treated as absent. The conditions

$$\left.\begin{aligned} \psi_1 &= \psi_2 \\ \frac{d\psi_1}{dr} &= \frac{d\psi_2}{dr} \end{aligned}\right\} \text{when } r = a$$

give

$$A_1 = 0, \qquad A_n = 0 \text{ when } n > 2,$$

$$B_1 = 0, \qquad B_n = 0 \text{ when } n > 2,$$

and for $n = 2$

$$\left.\begin{aligned} A_2 a^2 - \tfrac{2}{5}\pi k a^4 &= \frac{B_2}{a} \\ 2\,A_2\,a - \tfrac{8}{5}\pi k a^3 &= -\frac{B_2}{a^2} \end{aligned}\right\}$$

Hence

$$A_2 = \tfrac{2}{3}\pi k a^2, \qquad B_2 = \tfrac{4}{15}\pi k a^5,$$

and

$$\psi_1 = 2\pi k\left(\tfrac{1}{3}a^2 r^2 - \tfrac{1}{5}r^4\right)\sin^2\theta$$

$$\psi_2 = \tfrac{4}{15}\pi k \frac{a^5}{r}\sin^2\theta.$$

The velocity along the normal to the aggregate is

$$\frac{1}{2\pi\rho}\frac{d\psi_2}{r d\theta} = \tfrac{4}{15}ka^2\cos\theta.$$

Hence the aggregate moves forward through the surrounding fluid with a velocity

$$V = \tfrac{4}{15}ka^2.$$

Referred to the aggregate at rest therefore

$$\psi_1 = \tfrac{2}{5}\pi k r^2\left(a^2 - r^2\right)\sin^2\theta.$$

The cyclic constant (μ) is the circulation taken round a meridian section, up the polar axis and down outside. It is the sum of the circulation round the elementary areas of which the section is composed. Hence

$$\mu = \Sigma \text{ (elementary circulations)} = \Sigma 2\omega\, d\text{A} = k\Sigma 2\rho\, d\text{A}$$

$$= \frac{k}{\pi}\Sigma 2\pi\rho\, d\text{A} = \frac{k}{\pi} \times \text{volume of aggregate} = \frac{mk}{\pi}\cdot$$

Thus

$$\omega = k\rho = \frac{\pi\mu}{m}\rho, \qquad \text{V} = \frac{\mu}{5a},$$

which are HILL's results obtained by direct methods.

7. *Heterogeneous Aggregates.*—We may, however, superpose on an aggregate such as the foregoing other spherical layers of different vorticities. It will be advisable to consider first the case where there is one such layer of vorticity determined by (say) k'. We may call them dyads. In this outer portion both terms in $\text{A}r^n$ and B/r^{n-1} can appear. Let ψ_1, ψ_2, ψ denote the stream functions for each part and for the surrounding fluid. Then

$$\psi_1 = -\tfrac{2}{5}\pi k r^4 \text{Z}_2 + \Sigma \text{A}_n r^n \text{Z}_n,$$

$$\psi_2 = -\tfrac{2}{5}\pi k' r^4 \text{Z}_2 + \Sigma\left(\text{A}_n' r^n + \frac{\text{B}'_n}{r^{n-1}}\right)\text{Z}_n,$$

$$\psi = \Sigma \frac{\text{B}_n}{r^{n-1}}\text{Z}_n.$$

Let a, b denote the radii of the two spherical surfaces ($a > b$), and apply the same conditions as before to the two surfaces.

Again all the co-efficients vanish except for $n = 2$, and there results

$$\left.\begin{aligned} \text{A}_2 b^2 - \tfrac{2}{5}\pi k b^4 &= \text{A}_2' b^2 + \frac{\text{B}_2'}{b} - \tfrac{2}{5}\pi k' b^4 \\ 2\text{A}_2 b - \tfrac{8}{5}\pi k b^3 &= 2\text{A}_2' b - \frac{\text{B}_2'}{b^2} - \tfrac{8}{5}\pi k' b^3 \end{aligned}\right\}$$

and

$$\left.\begin{aligned} \frac{\text{B}_2}{a} &= \text{A}_2' a^2 + \frac{\text{B}_2'}{a} - \tfrac{2}{5}\pi k' a^4 \\ -\frac{\text{B}_2}{a^2} &= 2\text{A}_2' a - \frac{\text{B}_2'}{a^2} - \tfrac{8}{5}\pi k' a^3 \end{aligned}\right\}$$

The first two give at once

$$\text{B}_2' = \tfrac{4}{15}\pi (k - k')\, b^5,$$

the last two

$$\text{A}_2' = \tfrac{2}{3}\pi a^2 k' ;$$

also

$$\text{B}_2 = \tfrac{4}{15}\pi \{(k - k')\, b^5 + k' a^5\}$$

$$\text{A}_2 = \tfrac{2}{3}\pi \{k' a^2 + (k - k')\, b^2\}.$$

Whence

$$\psi_1 = 2\pi\left\{\frac{k'a^2 + (k-k')\,b^2}{3}\,r^2 - \tfrac{1}{5}kr^4\right\}\sin^2\theta,$$

$$\psi_2 = 2\pi\left\{\tfrac{1}{3}k'a^2r^2 + \tfrac{2}{15}(k-k')\,\frac{b^5}{r} - \tfrac{1}{5}k'r^4\right\}\sin^2\theta,$$

$$\psi = \frac{4\pi}{15}\,\frac{(k-k')\,b^5 + k'a^5}{r}\,\sin^2\theta.$$

The normal velocity at the outer boundary is

$$\frac{1}{2\pi\rho}\,\frac{d\psi}{rd\theta}\ (\text{when } r = a) = \tfrac{4}{15}\,\frac{(k-k')\,b^5 + k'a^5}{a^3}\cos\theta.$$

The outer boundary therefore progresses unchanged with velocity of translation

$$\mathrm{V} = \tfrac{4}{15}\,\frac{(k-k')\,b^5 + k'a^5}{a^3}\,.$$

Bring the outer boundary to rest by impressing on every part of the fluid a velocity equal and opposite to this, *i.e.*, adding to the stream-functions a term

$$-\tfrac{4}{15}\pi\,\frac{(k-k')\,b^5 + k'a^5}{a^3}\,r^2\sin^2\theta.$$

The relative motions are then given by

$$\psi_1 = \frac{2\pi}{5}\left\{\tfrac{1}{3}\left(5k'a^2 + 5(k-k')\,b^2 - 2k'a^2 - 2(k-k')\,\frac{b^5}{a^3}\right) - kr^2\right\}r^2\sin^2\theta$$

$$= \frac{2\pi}{5}\left\{k'a^2 - kr^2 + \tfrac{1}{3}(k-k')\left(5 - \frac{2b^3}{a^3}\right)b^2\right\}r^2\sin^2\theta$$

$$\psi_2 = \frac{2\pi}{5}\left\{k'(a^2 - r^2)\,r^2 + \tfrac{2}{3}(k-k')\,\frac{b^5}{a^3r}(a^3 - r^3)\right\}\sin^2\theta.$$

If, however, the motion is to be steady, the inner sphere must now be at rest, that is $\psi_1 = 0$ when $r = b$. We get, therefore, the following necessary relation between k, k', a, b,

$$k'a^2 - kb^2 + \tfrac{1}{3}(k-k')\left(5 - \frac{2b^3}{a^3}\right)b^2 = 0.$$

This may be written

$$2b^2k\,(a^3 - b^3) + k'\{3a^3(a^2 - b^2) - 2b^2(a^3 - b^3)\} = 0.$$

Both the expressions in the brackets are positive, hence k/k' must be negative or the rotations in opposite directions in the two portions.

Denote the cyclic constants of the inner and outer portions by μ_1, μ_2. As before, we see that they are respectively

$$\frac{k}{\pi} \times \text{vol.}$$

That is

$$\mu_1 = \frac{mk}{\pi} = \tfrac{4}{3}\, b^3 k,$$

$$\mu_2 = \frac{m'k'}{\pi} = \tfrac{4}{3}\,(a^3 - b^3)\, k'.$$

Substituting for k, k' in terms of μ_1, μ_2

$$\mathrm{V} = \tfrac{1}{5}\left\{\mu_1 \frac{b^2}{a^3} + \mu_2 \frac{a^5 - b^5}{a^3(a^3 - b^3)}\right\}.$$

The result is that a double aggregate is possible. If, however, the size is given the ratio of the vorticities must have a special value, and *vice versâ.* In terms of the radii it may be shown that

$$\mathrm{V} = -\frac{4(k - k')\, b^2 (a - b)\{2a^3 + 3b^3 + 4a^2 b + 6ab^2\}}{45a^3(a + b)}.$$

Three cases specially invite attention, (1) equal volumes, (2) both parts made of similar matter, *i.e.*, vorticities equal, and (3) equal cyclic constants.

Case i.—Here $a^3 = 2b^3$.

$$\frac{k'}{k} = -\frac{2b^5}{6b^3(a^2 - b^2) - 2b^5} = -\frac{1}{3 \times 2^{2/3} - 4},$$

$$\frac{k}{k'} = -\,{\cdot}76220 = -\tfrac{3}{4} \text{ nearly}.$$

Case ii.—$k' = -k$.

$$3a^3(a^2 - b^2) - 4b^2(a^3 - b^3) = 0.$$

Put $a/b \equiv x$, we get

$$3x^4 + 3x^3 - 4x^2 - 4x - 4 = 0.$$

This has three negative roots; the positive one is

$$x = 1{\cdot}3283 \quad \text{or} \quad \frac{a}{b} = \tfrac{4}{3} \text{ nearly}.$$

Case iii.—$\mu = -\mu'$ or

$$\frac{k}{a^3 - b^3} = -\frac{k'}{b^3} = \frac{k - k'}{a^3},$$

whence

$$-a^2b^3 - b^2(a^3 - b^3) + \tfrac{1}{3}a^3\left(5 - \frac{2b^3}{a^3}\right)b^2 = 0,$$

$$2a^3 + b^3 - 3a^2b = (a - b)(2a^2 - ab - b^2) = (a - b)^2(2a + b) = 0.$$

Equal circulations are therefore impossible.

8. *Polyads.*—Passing on now to the consideration of any number of layers, let the radii of the spherical boundaries from the inside outwards be denoted by $a_1, a_2 \ldots a_n$; the vorticities by $k_1, k_2 \ldots k_n$, and the stream-functions by $\psi_1, \psi_2 \ldots \psi_n$ and ψ. Then

$$\left.\begin{aligned} \psi_1 &= 2\pi \{ A_1 r^2 - \tfrac{1}{5} k_1 r^4 \} \sin^2\theta \\ \psi_p &= 2\pi \left\{ A_p r^2 + \frac{B_p}{r} - \tfrac{1}{5} k_p r^4 \right\} \sin^2\theta \\ \psi_{n+1} &= 2\pi \frac{B_{n+1}}{r} \sin^2\theta \end{aligned}\right\}.$$

Applying the conditions of continuity at the pth boundary, there results

$$A_p a_p^2 + \frac{B_p}{a_p} - \tfrac{1}{5} k_p a_p^4 = A_{p+1} a_p^2 + \frac{B_{p+1}}{a_p} - \tfrac{1}{5} k_{p+1} a_p^4,$$

$$2A_p a_p^2 - \frac{B_p}{a_p} - \tfrac{4}{5} k_p a_p^4 = 2A_{p+1} a_p^2 - \frac{B_{p+1}}{a_p} - \tfrac{4}{5} k_{p+1} a_p^4,$$

with

$$B_1 = 0, \qquad A_{n+1} = 0.$$

Adding

$$A_{p+1} - A_p = \tfrac{1}{3} (k_{p+1} - k_p)\, a_p^2 \quad \text{with} \quad A_{n+1} = 0.$$

Similarly

$$B_{p+1} - B_p = -\tfrac{2}{15} (k_{p+1} - k_p)\, a_n^5 \quad \text{with} \quad B_1 = 0.$$

Clearly the A's evolve from the outside, the B's from inside.

Write

$$\tfrac{1}{3} (k_p - k_{p+1}) = \lambda_p.$$

Then

$$A_p - A_{p+1} = \lambda_p a_p^2 \quad \text{with} \quad A_{n+1} = 0,$$

$$B_{p+1} - B_p = \tfrac{2}{5} \lambda_p a_p^5 \quad \text{with} \quad B_1 = 0.$$

Hence

$$A_p = \Sigma_p^n \lambda_p a_p^2, \qquad B_p = \tfrac{2}{5} \Sigma_1^{p-1} \lambda_p a_p^5.$$

Thus the ψ are completely determined.

For steadiness of motion it is necessary that the translatory velocity of the different boundaries be the same. This is obtained if the velocities of the inner and outer boundaries of each layer are equal.

Hence we get $n - 1$ equations ($p = 2$ to n)

$$\tfrac{1}{2} V = A_p + \frac{B_p}{a_p^3} - \tfrac{1}{5} k_p a_p^2 = A_p + \frac{B_p}{a_{p-1}^3} - \tfrac{1}{5} k_p a_{p-1}^2,$$

or

$$B_p \left(\frac{1}{a_p^3} - \frac{1}{a_{p-1}^3} \right) = -\tfrac{1}{5} k_p (a_{p-1}^2 - a_p^2),$$

or

$$B_p = -\tfrac{1}{5} k_p a_p^3 a_{p-1}^3 \frac{a_p^2 - a_{p-1}^2}{a_p^3 - a_{p-}^3}.$$

If the volumes of all the layers are equal,

$$a_p^3 - a_{p-1}^3 = a_1^3 \quad \text{and} \quad a_p^3 = pa_1^3.$$

Hence

$$\mathrm{B}_p = -\tfrac{1}{5}k_p p(p-1)\{p^{2/3}-(p-1)^{2/3}\}a_1^5$$

or

$$p(p-1)\{p^{2/3}-(p-1)^{2/3}\}k_p = -2\{(p-1)^{5/3}\lambda_{p-1}+\ldots\ldots+\lambda_1\}.$$

Now

$$\lambda_{p-1} = \tfrac{1}{3}(k_{p-1}-k_p).$$

Hence

$$\{(p-1)p^{5/3}-(p+\tfrac{2}{3})(p-1)^{5/3}\}k_p = -\tfrac{2}{3}(p-1)^{5/3}k_{p-1} - 2\{(p-2)^{5/3}\lambda_{p-2}+\ldots\ldots+\lambda_1\}$$
$$= -\tfrac{2}{3}\{((p-1)^{5/3}-(p-2)^{5/3})k_{p-1}+\ldots\ldots+(2^{5/3}-1)k_2+k_1\},$$

or subtracting two consecutive equations

$$\{p^{5/3}-(p+\tfrac{2}{3})(p-1)^{2/3}\}k_p = -\{(p-2)^{5/3}-(p-\tfrac{8}{3})(p-1)^{2/3}\}k_{p-1}.$$

Thus the k can be determined in order from the inside. The peculiarity is that the process can stop at any point. That is that if we have two poly-ads, with m and n layers respectively $(m > n)$ then the first n layers in the first will be precisely similar to those in the second. The values are

$$k_2 = -\frac{1}{3\times 2^{2/3}-4}k_1 = -1\cdot3120k_1$$
$$k_3 = +1\cdot4717k_1$$
$$k_4 = -1\cdot5866k_1$$

and when p is large

$$k_p = -k_{p-1}.$$

As another example, take the case where the layers are formed of the same material, *i.e.*, the vorticities alternately equal and opposite. Then $k_p = (-)^{p-1}k_1$

$$\lambda_p = \tfrac{2}{3}k_p = \tfrac{2}{3}k_1(-)^{p-1} \quad \text{but} \quad \lambda_n = \tfrac{1}{3}k_n = (-)^{n-1}\tfrac{1}{3}k_1$$

$$\tfrac{4}{15}(a_1^5 - a_2^5 + \ldots\ldots + \overline{-}|^{p-2}a_{p-1}^5) = \tfrac{1}{5}(-)^{p-2}a_p^3a_{p-1}^3\frac{a_p^2-a_{p-1}^2}{a_p^3-a_{p-1}^3}.$$

Let x_p denote the ratio a_{p+1}/a_p.

These values are then given by

$$x_p^3\frac{x_p^2-1}{x_p^3-1} = \tfrac{4}{3}\left\{1-\frac{1}{x_{p-1}^5}+\frac{1}{x_{p-1}^5x_{p-2}^5}-\ldots\ldots\right\}$$

and may be found in succession. The equations are, if b_p denote $1-\dfrac{1}{x_{p-1}^5}+\ldots$

$$x_p^3(x_p+1) = \tfrac{4}{3}b_p(x_p^2+x_p+1).$$

In which it is clear that

$$b_p = 1 - \frac{b_{p-1}}{x_{p-1}^5}.$$

If $b_p = \frac{1}{2}$, the equation is

$$x^3(x+1) = \tfrac{2}{3}(x^2 + x + 1),$$

the positive root of which is $x = 1$. In this case

$$b_{p+1} = 1 - \frac{\frac{1}{2}}{1} = \tfrac{1}{2}.$$

If ever b_p is nearly $\frac{1}{2} = \frac{1}{2} + \alpha$ (say), x_p is nearly $1 = 1 + \xi$ (say). Then, regarding α and ξ of same order

$$(1 + 3\xi + 3\xi^2)(2 + \xi) = \tfrac{2}{3}(3 + 3\xi + \xi^2)(1 + 2\alpha)$$

$$\xi = \tfrac{4}{5}\alpha + \tfrac{4}{5}\alpha\xi - \tfrac{5}{3}\xi^2 = \tfrac{4}{5}\alpha - \tfrac{32}{75}\alpha^2 = \tfrac{4}{5}\alpha(1 - \tfrac{8}{15}\alpha)$$

and

$$b_{p+1} = 1 - \frac{\frac{1}{2} + \alpha}{\{1 + \frac{4}{5}\alpha - \frac{32}{75}\alpha^2\}^5} = 1 - (\tfrac{1}{2} + \alpha)(1 - 4\alpha + \tfrac{176}{15}\alpha^2) = \tfrac{1}{2} + \alpha - \tfrac{28}{15}\alpha^2.$$

Hence b_p continually converges to $\frac{1}{2}$ and the value of x_p to 1 as p increases.

The first seven values are

$$\begin{array}{ll} x_1 = 1{\cdot}3283, & b_2 = {\cdot}7582. \\ x_2 = 1{\cdot}1840, & b_3 = {\cdot}6741. \\ x_3 = 1{\cdot}1284, & b_4 = {\cdot}6315. \\ x_4 = 1{\cdot}0987, & b_5 = {\cdot}6056. \\ x_5 = 1{\cdot}0802, & b_6 = {\cdot}5882. \\ x_6 = 1{\cdot}0674, & b_7 = {\cdot}5753. \\ x_7 = 1{\cdot}0580, & b_8 = {\cdot}5660. \end{array}$$

The succeeding values will be given to four figures by the foregoing approximations.

The velocities of translation of the series of aggregates are

Monad $V_1 = \frac{4}{15}k_1a_1^2 \qquad = V_1.$

Dyad $V_2 = \frac{1}{2}(7 - 5x_1^2)V_1 \qquad = -{\cdot}9110\,V_1.$

Triad $V_3 = \frac{1}{2}(7 - 10x_1^2 + 5x_1^2x_2^2)V_1 \qquad = +{\cdot}8615\,V_1.$

4-ad $V_4 = \frac{1}{2}(7 - 10x_1^2 + 10x_1^2x_2^2 - 5x_1^2x_2^2x_3^2)V_1 \qquad = -{\cdot}8282\,V_1.$

5-ad $V_5 = \frac{1}{2}(7 - 10x_1^2 + 10x_1^2x_2^2 - 10x_1^2x_2^2x_3^2 + 5x_1^2x_2^2x_3^2x_4^2)V_1 = \quad {\cdot}8023\,V_1.$

&c. $V_6 \qquad = -{\cdot}7833\,V_1.$

$V_7 \qquad = \quad {\cdot}7618\,V_1.$

$V_8 \qquad = -{\cdot}7462\,V_1.$

9. The form of the stream lines for a monad aggregate have been delineated by HILL. The *general* form of the stream lines for a poly-ad is obvious, and there is no special reason for drawing them accurately at present. It will be well, however, to determine the position of the equatorial axes, for the particular case of homogeneous poly-ads, that is in which $k_p = (-)^p k_1$.

The condition at an equatorial axis is that

$$\frac{1}{2\pi\rho}\frac{d\psi}{dr} = 0 \quad \text{when} \quad \theta = \frac{\pi}{2},$$

in which ψ denotes the stream-function referred to the boundary at rest. Applying this to the p-th layer in an n-ad

$$\psi_p = 2\pi\left\{\mathrm{A}_p r^2 + \frac{\mathrm{B}_p}{r} - \tfrac{1}{5}k_p r^4 - \mathrm{A}_p r^2 - \frac{\mathrm{B}_p r^2}{a_p^3} + \tfrac{1}{5}k_p a_p^2 r^2\right\}\sin^2\theta.$$

The equation for the equatorial axis is therefore

$$-\mathrm{B}_p\left(\frac{1}{r^3} + \frac{2}{a_p^3}\right) - \tfrac{4}{5}k_p r^2 + \tfrac{2}{5}k_p a_p^2 = 0,$$

or

$$r^5(a_p^3 - a_{p-1}^3) - \tfrac{1}{2}(a_p^5 - a_{p-1}^5)r^3 - \tfrac{1}{4}a_p^3 a_{p-1}^3(a_p^2 - a_{p-1}^2) = 0.$$

This may be written

$$r^5 - \tfrac{1}{2}\cdot\frac{x_{p-1}^5 - 1}{x_{p-1}^3 - 1}a_{p-1}^2 r^3 - \tfrac{1}{4}\frac{x_{p-1}^2 - 1}{x_{p-1}^3 - 1}x_{p-1}^3 a_{p-1}^5 = 0.$$

Now

$$x_{p-1}^3(x_{p-1}^2 - 1) = \tfrac{4}{3}b_{p-1}(x_{p-1}^3 - 1),$$

therefore

$$\frac{x_{p-1}^5 - 1}{x_{p-1}^3 - 1} - 1 = \tfrac{4}{3}b_{p-1},$$

and

$$r^5 - \tfrac{1}{2}(\tfrac{4}{3}b_{p-1} + 1)a_{p-1}^2 r^3 - \tfrac{1}{3}b_{p-1}a_{p-1}^5 = 0.$$

For a monad

$$b = 0 \qquad r^2 = \frac{a^2}{2} \qquad r = \frac{a}{\sqrt{2}},$$

for a dyad

$$b_1 = 1 \qquad r^5 - \tfrac{7}{6}a_1^2 r^3 - \tfrac{1}{3}a_1^5 = 0 \qquad r = 1{\cdot}1720a_1.$$

Beyond dyads

$$r = a_{p-1} \text{ nearly} = (1 + \xi)a_{p-1}.$$

Then

$$a_{p-1}^5\{1 - \tfrac{1}{2}(\tfrac{4}{3}b_{p-1} + 1) - \tfrac{1}{3}b_{p-1}\} + a_{p-1}^5\xi\{5 - \tfrac{3}{2}(\tfrac{4}{3}b_{p-1} + 1)\} = 0.$$

Now b_{p-1} is nearly $\frac{1}{2}$

$$= \tfrac{1}{2} + f_{p-1} \qquad (\tfrac{5}{2} - 2f_{p-1})\xi = f_{p-1} \qquad \xi = \frac{2f_{p-1}}{5} \qquad r_p = \left(1 + \frac{2f_{p-1}}{5}\right)a_{p-1}.$$

The distance from the inner layer is therefore

$$r_p - a_{p-1} = \tfrac{2}{5} f_{p-1} a_{p-1}.$$

From the outer it is

$$a_p - r = a_p - a_{p-1} - \tfrac{2}{5} f_{p-1} a_{p-1}$$

$$\text{Ratio} = \frac{\tfrac{2}{5} f_{p-1}}{x_{p-1} - 1 - \tfrac{2}{5} f_{p-1}}$$

But (p. 50)

$$x_{p-1} = 1 + \tfrac{4}{5} f_{p-1},$$

therefore

$$\text{Ratio} = 1,$$

or the equatorial axis, with increasing number of layers, tends to bisect the distance between the two boundaries of the layer.

10. *Energy.*—The energy within any region is

$$\mathrm{E} = \tfrac{1}{2} \iint \frac{1}{(2\pi\rho)^2} \left\{ \left(\frac{d\psi}{d\rho}\right)^2 + \left(\frac{d\psi}{dz}\right)^2 \right\} 2\pi\rho \, d\rho \, dz,$$

the integral extending within the boundary of the region. By the ordinary method this is reduced to the form

$$\mathrm{E} = -\frac{1}{4\pi} \int \frac{\psi}{\rho} \frac{d\psi}{dn} \, ds + \iint \omega\psi \, d\rho \, dz.$$

Since

$$\frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d\psi}{d\rho}\right) + \frac{d}{dz}\left(\frac{1}{\rho}\frac{d\psi}{dz}\right) = -4\pi\omega.$$

If the boundary be infinite and the fluid at rest then the first integral is zero, and

$$\mathrm{E} = \iint \omega\psi \, d\rho \, dz.$$

The integral extending only to spaces which contain rotational motion. If the motion is of uniform vorticity $\omega = k\rho$, and

$$\mathrm{E} = k \iint \rho\psi \, d\rho \, dz.$$

In the cases here considered ψ is of the form $f(r) \sin^2 \theta$, and

$$\mathrm{E} = 2k \iint_0^{\frac{\pi}{2}} r^2 f(r) \, dr \sin^3 \theta \, d\theta = \tfrac{4}{3} k \int r^2 f(r) \, dr.$$

In the case of a poly-ad $f(r)$ is different for the various layers, and

$$\mathrm{E} = \tfrac{4}{3} \left\{ k_1 \int_0^{a_1} r^2 f_1(r) \, dr + k_2 \int_{a_1}^{a_2} r^2 f_2(r) \, dr + \ldots \right\}.$$

We work out the case for a dyad aggregate, in which $k_2 = -k_1$,

$$f_1(r) \equiv 2\pi\{(\tfrac{2}{3}a_1^2 - \tfrac{1}{3}a_2^2)k_1r^2 - \tfrac{1}{5}k_1r^4\},$$
$$f_2(r) \equiv 2\pi\{-\tfrac{1}{3}k_1a_2^2r^2 + \tfrac{4}{15}k_1a_1^5/r + \tfrac{1}{5}k_1r^4\},$$

and

$$\begin{aligned} \mathrm{E} &= \frac{8\pi k_1^2}{3}\{\tfrac{1}{15}(2a_1^2 - a_2^2)a_1^5 - \tfrac{1}{5.7}a_1^7 + \tfrac{1}{15}a_2^2(a_2^5 - a_1^5) \\ &\qquad - \tfrac{2}{15}a_1^5(a_2^2 - a_1^2) - \tfrac{1}{5.7}(a_2^7 - a_1^7)\} \\ &= \frac{8\pi k_1^2}{15}\{\tfrac{4}{3}a_1^7 - \tfrac{4}{3}a_2^2a_1^5 + \tfrac{4}{3.7}a_2^7\} \\ &= \frac{32\pi k_1^2}{45}(a_1^7 - a_2^2a_1^5 + \tfrac{1}{7}a_2^7) \\ &= 1{\cdot}945 \times \frac{32\pi k_1^2}{45 \times 7}a_1^7. \end{aligned}$$

If the two parts had been single monads their combined energy (when far apart) would have been

$$\mathrm{E} = 1{\cdot}534 \times \frac{32\pi k_1^2}{45 \times 7}a_1^7.$$

The energy when combined is therefore greater than when they are separate.

11. It may not be out of place to make a short digression here as to the relation of a HILL's vortex to the vortex rings which have been investigated in previous parts of these researches. As is known the translation velocity of an ordinary ring decreases as the energy increases, and formulæ are given in a former paper* whereby those quantities can be calculated for comparatively thick rings up to $\mathrm{R}/r = 4$ with considerable accuracy, and possibly further. Here R is the radius of the equatorial axis and r the mean radius of the section of the ring. Refer all measurements to the spherical form, and let c denote its radius, V_0 its velocity of translation, and E_0 its energy. Take now a ring of the same volume and circulation as the sphere, and let V and E denote its translation velocity and energy. We get the following value of E/E_0, V/V_0 for different apertures.

$\frac{\mathrm{R}}{r}$.	$\frac{\mathrm{R}}{c}$.	$\frac{r}{c}$.	$\frac{\mathrm{V}}{\mathrm{V}_0}$.	$\frac{\mathrm{E}}{\mathrm{E}_0}$.
100	..	..	·199	176
50	8·09	·162	·282	95
10	2·77	·277	·593	20·8
5	1·745	·349	·784	10·25
4	1·500	·375	·856	8
3	1·239	·413	*·946*	*6*

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* "Researches in the Theory of Vortex Rings," Part II., p. 757, 'Phil. Trans.,' 1885, Part II.

These numbers are graphically represented in fig. 1, Plate 1, where the abscissæ give E/E_0 and the ordinates V/V_0. Dotted lines refer to points where calculation cannot be applied. On the same figure are placed outlines of the aggregates drawn to scale. Two things at once strike the eye. First, that the spherical aggregate evidently lies on the E.V curve of the rings, belongs, in fact, to the same family; and, secondly, that the variation of V with the energy is small over a very large range. The shape and nature of the aggregate when the energy is nearly that of the spherical form have not yet been determined. It is probable that as the energy diminishes the form lengthens along the polar axis, until when the energy is very small it becomes a long, thin, cylindrical aggregate. When this is so long that the end portions form only a small portion of the whole, it is possible to obtain an approximation to the energy, for when very long the fluid outside will be very nearly at rest (as in case of force outside a long helix). The velocity of propagation will then be the velocity at the axis. Let a be the radius of the cylinder, l its length. Then

$$la^2 = \tfrac{4}{3}c^3.$$

Again, if V denote the velocity along the axis, the velocity outside is zero, and the variation at the ends only a small part of the whole. Hence the circulation is given by

$$\mu = Vl.$$

Again let v denote the velocity at a distance r from the axis. Take a small rectangular circuit, b parallel to the axis, one inside distant r from the axis, the other outside. The circulation round this is bv. But it is also the value $\Sigma\omega d\mathrm{A}$ taken over the area of the rectangle.

Therefore

$$bv = k\Sigma 2r\,d\mathrm{A} = \frac{k}{\pi}\,(\text{volume}) = \frac{k}{\pi}\,.\,b\pi\,(a^2 - r^2),$$

$$v = k\,(a^2 - r^2), \qquad \mu = lka^2\,;$$

therefore

$$v = \frac{\mu}{l}\left(1 - \frac{r^2}{a^2}\right).$$

$$\text{Energy in E} = \int_0^r 2\pi r\,.\,l\,dr\,.\,\tfrac{1}{2}v^2$$

$$= \frac{\pi\mu^2}{2l}\int_0^r\left(1 - \frac{r^2}{a^2}\right)^2 d\,(r^2)$$

$$= \frac{\pi\mu^2 a^2}{6l} = \frac{\mu^2}{6l^2}\,.\,m = \tfrac{2}{9}\pi\mu^2\,\frac{c}{l}$$

$$= \tfrac{1}{6}\,m\mathrm{V}^2 = \tfrac{2}{9}\,\pi c^3\mathrm{V}^2.$$

It is thus the same as a mass of one-third its own mass moving with its velocity of translation. Now

$$E_0 = \tfrac{2}{35}\pi\mu^2 c\,, \qquad V_0 = \frac{\mu}{5c}\,,$$

therefore

$$\frac{E}{E_0} = \tfrac{35}{9}\frac{c^2}{l^2}\,, \qquad \frac{V}{V_0} = \frac{5c}{l}\,,$$

therefore

$$\frac{E}{E_0} = \tfrac{7}{45}\left(\frac{V}{V_0}\right)^2.$$

This only holds, however, when V/V_0 is small. It is a small part of a parabola in the figure touching the axis of E/E_0.

12. *Spheroidal Aggregates.*—As is known from HILL's investigations, the spheroid, although an instantaneously possible form, is not steady. It proceeds at once to change its shape into a non-spheroidal one. It seems, however, advisable to give the general outline of the method as adopted in this paper and as applied to the spheroids, in order to investigate whether by superposing a second or third layer it may be possible to obtain a steady form.

The functions involved and the differential equation for ψ are given in Eqs. 8, 9. Writing C for $\cosh u$ and S for $\sinh u$, the differential equation in ψ is

$$\frac{1}{\cos v}\frac{d}{du}\left(\frac{1}{S}\frac{d\psi}{du}\right) + \frac{1}{S}\frac{d}{dv}\left(\frac{1}{\cos v}\frac{d\psi}{dv}\right) = -\,4\pi k\lambda^4 S\cos v\,(C^2 - \sin^2 v),$$

since

$$\omega = k\rho = k\lambda S\cos v.$$

As in the former case, a particular integral is

$$\psi = -\,\frac{\pi}{2}k\rho^4 = -\,\tfrac{1}{2}\pi k\lambda^4 S^4\cos^4 v.$$

It remains to integrate

$$\frac{1}{\cos v}\frac{d}{du}\left(\frac{1}{S}\frac{d\psi}{du}\right) + \frac{1}{S}\frac{d}{dv}\left(\frac{1}{\cos v}\frac{d\psi}{dv}\right) = 0.$$

This can be satisfied by writing $\psi = \Sigma X_m Z_m$ where X and Z are functions respectively of u and v only, and

$$\left.\begin{aligned} \frac{d}{dv}\left(\frac{1}{\cos v}\frac{dZ}{dv}\right) &= -\,\frac{mZ}{\cos v} \\ \frac{d}{du}\left(\frac{1}{S}\frac{dX}{du}\right) &= \frac{mX}{S} \end{aligned}\right\},$$

m being any constant. These equations are

$$\left.\begin{aligned}\frac{d^2Z}{dv} + \tan v\frac{dZ}{dv} + mZ = 0\\ \frac{d^2X}{du^2} - \coth u\frac{dX}{du} - mX = 0\end{aligned}\right\}.$$

As will be seen later m must be of the form $n(n-1)$, n being any integer. Writing for a moment $v = \frac{\pi}{2} - \theta$, the equation in Z becomes

$$\frac{d^2Z}{d\theta^2} - \cot\theta\frac{dZ}{d\theta} + n(n-1)Z = 0,$$

whence

$$Z_n = -\sin\theta\frac{dP_{n-1}}{d\theta}.$$

Therefore

$$\begin{aligned}Z_2 &= \sin^2\theta = \cos^2 v\\ Z_4 &= 6\sin^2\theta - \tfrac{15}{2}\sin^4\theta\\ &= 6\cos^2 v - \tfrac{15}{2}\cos^4 v\\ \cos^4 v &= \tfrac{4}{5}Z_2 - \tfrac{2}{15}Z_4.\end{aligned}$$

To determine X, we proceed by the same analogy to put

$$X = S\frac{dP}{du}.$$

Then

$$\frac{d^2X}{du^2} - \coth u\frac{dX}{du} - n(n-1)X = S\frac{d}{du}\left\{\frac{d^2P}{du^2} + \coth u\frac{dP}{du} - n(n-1)P\right\}.$$

If then P denote a zonal harmonic with imaginary argument and of order $n-1$, the right hand of the above vanishes, and the value of X is a solution. That is

$$X_n = S\frac{dP_{n-1}}{du}.$$

Now we have

$$P_n = \frac{1.3\ldots(2n-1)}{n!}\left\{C^n - \frac{n(n-1)}{2(2n-1)}C^{n-2} + \ldots\right\}.$$

Hence

$$X_2 = S^2. \qquad X_4 = 6S^2 + \tfrac{15}{}S^4. \qquad S^4 = \tfrac{2}{15}X_4 - \tfrac{4}{5}X_2.$$

This set of solutions gives values finite and continuous at all points inside a given ellipse of the family, but infinitely large at an infinite distance. Let Y denote the second integral of the equation. Then, in the usual way, it may be shown that

$$Y = X\int\frac{S}{X^2}du,$$

whence it is easy to prove that

$$Y_2 = S^2 \int \frac{du}{S^3} = \tfrac{1}{4} S^2 \log \frac{C+1}{C-1} - \tfrac{1}{2} C,$$

$$Y_4 = \tfrac{1}{24} X_4 \log \frac{C+1}{C-1} - \tfrac{1}{24} C \, (15C^2 - 13).$$

With these values the particular integral is

$$\tfrac{2}{25} \pi k \lambda^4 (2X_2 - \tfrac{1}{3} X_4)(2Z_2 - \tfrac{1}{3} Z_4).$$

The terms in X_2Z_2, X_4Z_4 may be supposed merged in the general solution. We may then write

$$\psi_1 = (A_2X_2 - \tfrac{4}{75} \pi k \lambda^4 X_4) Z_2 + (A_4X_4 - \tfrac{4}{75} \pi k \lambda^4 X_2) Z_4,$$

$$\psi_2 = B_2Y_2Z_2 + B_4Y_4Z_4.$$

From these it is easy to deduce the values of A_2, etc., for a single free aggregate, by applying the conditions $\psi_1 = \psi_2$ and $d\psi_1/du = d\psi_2/du$ at the surface. It is unnecessary to do this, as from HILL's work we know that it is not steady.

The case of motion inside a rigid spheroidal boundary is also given by HILL.* The solution follows immediately by impressing the condition $\psi_1 = 0$ when $u = \mathbf{u}$.

Hence

$$A_2 = \tfrac{4}{75} \pi k \lambda^4 \frac{\mathbf{X}_4}{\mathbf{X}_2},$$

$$A_4 = \tfrac{4}{75} \pi k \lambda^4 \frac{\mathbf{X}_2}{\mathbf{X}_4},$$

where thick type denotes values at the surface, and

$$\psi_1 = \tfrac{4}{75} \pi k \lambda^4 \frac{X_2\mathbf{X}_4 - \mathbf{X}_2X_4}{\mathbf{X}_2} Z_2 - \tfrac{4}{75} \pi k \lambda^4 \, . \, \frac{X_2\mathbf{X}_4 - \mathbf{X}_2X_4}{\mathbf{X}_4} Z_4,$$

which easily reduces to

$$\psi_1 = \frac{2\pi k \lambda^4}{4 + 5\mathbf{S}^2} (\mathbf{S}^2 - S^2) \, S^2 (\mathbf{S}^2 + \cos^2 v) \cos^2 v.$$

The total circulation is $\frac{4}{3} k\lambda^3 \mathbf{CS}^2$.

The equatorial axis is given by

$$\frac{d\psi}{du} = 0, \quad \text{when } v = 0.$$

That is by the equation

$$2\mathbf{S}^2 S - 4S^3 = 0, \quad \text{or } S = \frac{1}{\sqrt{2}} \mathbf{S}.$$

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* 'Phil. Trans.' Part II., 1884, p. 403.

The equatorial axis therefore lies in the equatorial section in a similar position to that for a sphere.

13. *Dyad Spheroids.*—Poly-ad spheroids clearly occur in the same way as for spheres; they are, however, also unsteady. It will be sufficient merely to indicate the steps of the proof.

Let $u = u'$ and $u = u''$ denote the two boundaries. ψ will involve terms in Z_2 and Z_4. By applying the surface conditions in the same way as for the spheres to both sets of terms independently, the coefficients are determined, whilst the condition that the internal interface has the same translational velocity as the outer gives for Z_4 an equation which u' and u'' must satisfy. This is

$$\frac{C'S'^4}{S''} M'' - \frac{C''S''^4}{S'} M' + \frac{C''S''^4 - C'S'^4}{6 - \frac{15}{2} S'^2} = 0,$$

where

$$M = X_2 \frac{dY_4}{du} - Y_4 \frac{dX_2}{du},$$

and the dashed letters refer to values at the outer and inner boundaries u'', u'.

The same applied to the Z_2 terms give

$$\frac{C'S'^4}{S''} N'' - \frac{C''S''^4}{S'} N' + (C''S''^4 - C'S'^4) \frac{X_4'}{X_2'} = 0,$$

where

$$N = X_4 \frac{dY_2}{du} - Y_2 \frac{dX_4}{du}.$$

The existence of steadily-moving spheroids depends on the possibility of finding values of u', u'' to satisfy these two equations.

It is easy to show that

$$6M + N = \tfrac{1}{2} S (25S^2 + 14).$$

Hence, adding 6 times the first equation to the second, there results an equation free of logarithmic terms and which can easily be reduced to

$$\frac{S'^2 (S''^2 - S'^2)}{C''S''^4 - C'S'^4} = \frac{2C'}{5C'^2 - 1}.$$

Putting $C' = y$, $C'' = x$, the factor $(x - y)^2$ divides out, and the equation may be put in the form

$$2y(x^3 + 2x^2y + 3xy^2 - 2x) + (y^2 - 1)(3y^2 + 1) = 0.$$

Now $x > y > 1$. Hence $3xy^2 - 2x \equiv xy^2 + 2x(y^2 - 1)$ is positive. The expression on the left is therefore always positive and no suitable values of x, y satisfy the equation. A prolate spheroidal dyad is therefore not steady.

The condition for the oblate spheroid can be found by writing $S\sqrt{-1}$ for C. It can be shown that this also has no suitable root.

Section iii.—*Gyrostatic Aggregates.*

14. Passing on now to the consideration of the more general problem where a secondary spin exists, the simplest case is that in which in equation (12) both F and $f df/d\psi$ are uniform.

Suppose

$$f\frac{df}{d\psi} = \mathrm{A}, \quad \text{or} \quad f = \sqrt{(2\mathrm{A}\psi)}.$$

The differential equation in ψ is now

$$\frac{d^2\psi}{dr^2} + \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} - \frac{\cot\theta}{r^2}\frac{d\psi}{d\theta} = -8\pi^2\rho^2\mathrm{F} - \mathrm{A},$$

a particular integral of which is

$$\psi = -\pi^2\rho^4\mathrm{F} - \tfrac{1}{2}\mathrm{A}r^2,$$

and the general integral is the same as that considered in the previous section, viz.:

$$\left(\mathrm{A}r^n + \frac{\mathrm{B}}{r^{n-1}}\right)\mathrm{Z}_n.$$

It will however not be found possible to satisfy the boundary conditions unless the term $\psi = \mathrm{A}_1 r$ be introduced. This term, as well as that in $\frac{1}{2}\mathrm{A}r^2$, makes the motion discontinuous at the polar axis. However, we will suppose for the moment this portion of space excluded, and see later if it is possible to do so. The stream-functions are then,—inside

$$\psi_1 = -\pi^2\rho^4\mathrm{F} - \tfrac{1}{2}\mathrm{A}r^2 + \mathrm{A}_1 r + \Sigma_2\mathrm{A}_n r^n\mathrm{Z}_n,$$

outside

$$\psi_2 = \Sigma\frac{\mathrm{B}_n}{r^{n-1}}\mathrm{Z}_n,$$

and ρ^4 can be replaced as before by $\frac{4}{5}r^4\mathrm{Z}_2$.

Applying the conditions $\psi_1 = \psi_2$ and $d\psi_1/dr = d\psi_2/dr$, when $r = a$ it is easy to deduce that

$$\psi_1 = -\tfrac{1}{2}\mathrm{A}(a-r)^2 - \tfrac{4}{5}\pi^2\mathrm{F}r^4\mathrm{Z}_2 + \tfrac{4}{3}\pi^2a^2\mathrm{F}r^2\mathrm{Z}_2,$$

$$\psi_2 = \tfrac{8}{15}\pi^2\mathrm{F}\frac{a^5}{r}\mathrm{Z}_2.$$

The velocity normal to the sphere is

$$\left[\frac{1}{2\pi\rho}\frac{d\psi}{rd\theta}\right]_{r=a} = \tfrac{8}{15}\pi\mathrm{F}a^2\cos\theta.$$

That is, the sphere progresses bodily with a velocity given by

$$\mathrm{V} = \tfrac{8}{15}\pi a^2\mathrm{F}.$$

Impress $-$ V on every point, that is, deduct $\frac{8}{15}\pi^2 a^2 \mathrm{F} r^2 \mathrm{Z}_2$. Then the stream-function referred to the boundary is

$$\psi_1 = -\tfrac{1}{2}\mathrm{A}\,(a-r)^2 + \tfrac{4}{5}\,\pi^2 \mathrm{F} r^2 (a^2 - r^2)\,\mathrm{Z}_2.$$

At the outer boundary $\psi = 0$. If we trace the stream-line $\psi = 0$, it is seen that it consists of the circle $r = a$ and the curve

$$\tfrac{1}{2}\mathrm{A}\ (a - r) = \tfrac{4}{5}\pi^2 \mathrm{F} r^2\,(a + r)\sin^2\theta.$$

This passes through the poles ($r = a$, $\theta = 0$) and touches the circle there. Hence the space between this and the outer boundary does not contain the polar axis. The motion given by ψ is therefore finite and continuous there. The space inside it must be excluded as giving a motion not possible—or rather, a motion due to sources and sinks on the polar axis. We shall suppose it excluded by replacing the fluid by a solid nucleus of the shape required.

The radius of an equatorial axis is given by $d\psi/dr = 0$ when $\theta = \pi/2$, or by

$$\mathrm{A}\,(a - r) + \tfrac{8}{5}\pi^2 \mathrm{F} r\,(a^2 - 2r^2) = 0.$$

In this write $r/a = x$ and $\dfrac{5\mathrm{A}}{16\pi^2 \mathrm{F} a^2} = b$. Then

$$x^3 + (b - \tfrac{1}{2})\,x - b = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

This has one root between 0 and 1. The other roots must either be both imaginary, or, if real, one at least must be negative, since the coefficient of x^2 is zero. As, further, $x = -\infty$ and $x = 0$ both make the expression on the left of the same sign, both these roots must be negative. Hence there is one and only one root between 0 and 1. That is, there is only one equatorial axis.

In the special case $b = \frac{1}{2}$, the radius of the equatorial axis is $a \,.\, 2^{-\frac{1}{3}} = {\cdot}7937a$. For this curve

$$\psi_1 = \tfrac{1}{2}\mathrm{A}\left\{\frac{r^2}{a^2}\,(a^2 - r^2)\,\mathrm{Z}_2 - (a - r)^2\right\}.$$

The curves are drawn in fig. 1, Plate 2, for values of $2\psi/\mathrm{A}a^2 = -{\cdot}1,\ 0,\ +{\cdot}1$. The value at the equatorial axis is ·397. The value ($-$ ·1) is drawn to show how the discontinuity enters.

The velocity along a parallel of latitude is given by the equation

$$2\pi\rho v \sin\phi = f = \sqrt{(2\mathrm{A}\psi)}.$$

This is zero at the surface and on the spindle-shaped nucleus, and increases to a maximum at the equatorial axis. The secondary cyclic constant is the circulation round the two circles (1) the equator of the sphere, and (2) the equatorial axis. It is therefore given by

$$\nu = \sqrt{(2\text{A}\psi')},$$

or

$$\nu = \text{A}a\sqrt{(2x - 1 - x^4)}$$

where x is the root of equation (13).

On account of the artificial nature of the internal nucleus the further discussion of this case is scarcely called for. We pass on, therefore, to the more important case—the next simplest one—in which F is uniform, but the second terms varies as ψ.

15. *Case* $f\frac{df}{d\psi} \propto \psi$.—Here also f varies as ψ.

Write $f = \frac{\lambda}{a}\psi$ where a is a length, which may be taken to be the radius of the sphere, and λ is a pure number. Also write $\text{F} = \frac{8\pi^2a^2}{\text{V}}$ where V is a velocity. Then the equation in ψ is

$$\frac{d^2\psi}{dr^2} + \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} - \frac{\cot\theta}{r^2}\frac{d\psi}{d\theta} = -\frac{\rho^2}{a^2}\text{V} - \frac{\lambda^2}{a^2}\psi.$$

A particular integral is $-\frac{\text{V}}{\lambda^2}\rho^2$ and the general integral depends on

$$\frac{d^2\psi}{dr^2} + \frac{1}{r^2}\frac{d\psi}{d\theta^2} - \frac{\cot\theta}{r^2}\frac{d\psi}{d\theta} + \frac{\lambda^2}{a^2}\psi = 0.$$

In this put $\psi = \text{J}_n\text{Z}_n$ where Z_n is the function of θ already discussed (§ 6) and J_n is a function of r only. Then

$$\frac{d^2\text{J}_n}{dr^2} - \left\{\frac{n(n-1)}{r^2} - \frac{\lambda^2}{a^2}\right\}\text{J}_n = 0.$$

$\text{J}_n/\sqrt{r}$ is therefore a Bessel's function of order $n - \frac{1}{2}$, which can, as is known, be expressed in finite form involving circular functions. In what immediately follows, the values of J_2 will alone be required. The equation is, writing x for r/a, and dropping the subscript 2,

$$\frac{d^2\text{J}}{dx^2} - \left(\frac{2}{x^2} - \lambda^2\right)\text{J} = 0.$$

If J and Y denote the two integrals

$$\text{J} = \frac{\sin\lambda x}{\lambda x} - \cos\lambda x,$$

$$\text{Y} = \frac{\cos\lambda x}{\lambda x} + \sin\lambda x,$$

or, in more general terms,

$$\text{Integral} = \mathrm{C}\left\{\frac{\sin(\alpha + \lambda x)}{\lambda x} - \cos(\alpha + \lambda x)\right\}$$

where C and α are arbitrary constants.

J and Y may be expressed in infinite convergent series. Thus

$$\begin{aligned}\mathrm{J}(y) \equiv \frac{\sin y}{y} - \cos y &= \tfrac{1}{3}y^2 - \frac{1}{2.3.5}y^4 + \ldots (-)^{n+1}\frac{2n}{(2n+1)!}y^{2n} \\ &= \tfrac{1}{3}y^2\left\{1 - \frac{y^2}{10} + \ldots + (-)^m \frac{3}{2m+3}\frac{y^{2m}}{(2m+1)!} + \ldots\right\} \quad . \quad (14)\end{aligned}$$

$$\begin{aligned}\mathrm{Y}(y) \equiv \frac{\cos y}{y} + \sin y &= \frac{1}{y} + \tfrac{1}{2}y + \ldots + (-)^{n+1}\frac{2n-1}{(2n)!}y^{2n-1} + \ldots \\ &= \frac{1}{y}\left\{1 + \tfrac{1}{2}y^2 + \ldots + (-)^{n+1}\frac{2n-1}{(2n)!}y^{2n} + \ldots\right\} \quad . \quad . \quad (15),\end{aligned}$$

also,

$$\left.\begin{aligned}\frac{d\mathrm{J}(y)}{dy} &= \sin y - \frac{\mathrm{J}}{y} = \mathrm{Y} - \frac{\sin y}{y^2} \\ \frac{d\mathrm{Y}(y)}{dy} &= \cos y - \frac{\mathrm{Y}}{y} = -\mathrm{J} - \frac{\cos y}{y^2} \\ \mathrm{Y}\frac{d\mathrm{J}}{dy} &- \mathrm{J}\frac{d\mathrm{Y}}{dy} = 1\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

and

Clearly the functions J refer only to space excluding infinity; Y to space excluding the origin.

16. For the problem in question the stream-functions are, therefore,

inside,

$$\psi_1 = -\frac{\mathrm{V}}{\lambda^2}r^2\sin^2\theta + \Sigma \mathrm{A}_n\mathrm{J}_n\mathrm{Z}_n,$$

outside,

$$\psi_2 = \Sigma\frac{\mathrm{B}_n}{r^{n-1}}\mathrm{Z}_n.$$

Applying the surface conditions that when $x = 1$, $\psi_1 = \psi_2$, and $d\psi_1/dx = d\psi_2/dx$, it follows that when

$$n > 2, \qquad \mathrm{A}_n = \mathrm{B}_n = 0,$$

when

$$n = 2,$$

$$-\frac{\mathrm{V}}{\lambda^2}a^2 + \mathrm{A}_2\mathrm{J}' = \frac{\mathrm{B}_2}{a},$$

$$-\frac{2\mathrm{V}}{\lambda^2}a^2 + \mathrm{A}_2\frac{d\mathrm{J}'}{dx} = -\frac{\mathrm{B}_2}{a},$$

where J′ and $d\mathrm{J}'/dx$ mean the values of J and $d\mathrm{J}/dx$ when $x = 1$, that is

$$J' = \frac{\sin\lambda}{\lambda} - \cos\lambda,$$

$$\frac{dJ'}{dx} = \lambda\sin\lambda - \frac{\sin\lambda}{\lambda} + \cos\lambda = \lambda\sin\lambda - J',$$

the two equations for A_2, B_2 give

$$A_2 = \frac{3V}{\lambda^3\sin\lambda}a^2, \qquad B_2 = \frac{Va^3}{\lambda^2}\left(\frac{3J'}{\lambda\sin\lambda} - 1\right).$$

The aggregate moves through the fluid with a velocity of translation given by

$$U = \frac{2B_2}{2\pi a^3} = \frac{V}{\pi\lambda^2}\left(\frac{3J'}{\lambda\sin\lambda} - 1\right).$$

By its formation the above value of ψ satisfies all the equations of condition except that in those equations ψ is the velocity-function referred to fixed axes. Here it is not—it represents the motion referred to the instantaneous position of the sphere. It is, therefore, not directly applicable unless the velocity of translation given by it vanishes, that is, unless

$$J' - \tfrac{1}{3}\lambda\sin\lambda = 0.$$

If λ be a root of this equation we get a steady motion of a vortex aggregate, at rest in the surrounding fluid.

If we, however, take the above general function, it gives a velocity of translation

$$U = \frac{V}{\pi\lambda^2}\left(\frac{3J'}{\lambda\sin\lambda} - 1\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

Bring the aggregate to rest by impressing a velocity—U on the whole fluid—that is, add to the stream-function a term $-\pi U\rho^2 = -\pi U a^2x^2\sin^2\theta$.

We get a new value of ψ, referred to axes remaining fixed, viz.,

$$\psi = \frac{3Va^2}{\lambda^3\sin\lambda}(J - x^2J')\sin^2\theta.$$

Take this value of ψ, and put $f = \frac{\lambda}{a}\psi$. Then equations (1, 3, 4, 7) become

$$v\rho\cos\phi = \frac{1}{2\pi}\frac{d\psi}{dn}$$

$$v\rho\sin\phi = \frac{\lambda}{2\pi a}\psi$$

$$\omega\rho\cos\chi = -\frac{\lambda}{4\pi a}\frac{d\psi}{dn}$$

$$\omega\rho\sin\chi = \frac{3V}{4\pi\lambda\sin\lambda}J\sin^2\theta.$$

These give v, ω, ϕ, χ.

Now substitute in $v\omega \sin \overline{\phi + \chi}$. The result is that

$$v\omega \sin (\phi + \chi)\, dn = \frac{3\mathrm{V}}{8\pi^2 a^2 \lambda \sin \lambda} \mathrm{J}'.d\psi$$

so that the motion given by the new ψ is a steady one. There exist, therefore, systems travelling through the fluid with velocities given by (17) and with a steady motion. The system given by $\mathrm{J}' = \frac{1}{3}\lambda \sin \lambda$ is contained as a special case.

17. There are two circulations to be considered. That along a circuit up the polar axis and down over the surface of the sphere, and that due to the motion round the polar axis. Call them respectively the primary and secondary cyclic constants, and denote them by μ, ν.

$$\mu = 2\int_0^a \left\{\frac{1}{2\pi\rho}\frac{d\psi}{r\,d\theta}\right\}_{\theta=0} dr + 2\int_0^{\pi/2}\left\{-\frac{1}{2\pi\rho}\frac{d\psi}{dr}\right\}_{r=a} a\,d\theta.$$

In finding this the term $x^2\mathrm{J}'\sin^2\theta$ may be omitted as giving no circulation, and we may take

$$\psi = \frac{3\mathrm{V}a^2}{\lambda^3 \sin \lambda} \mathrm{J} \sin^2\theta$$

$$\mu = \frac{3\mathrm{V}a^2}{\pi\lambda^3 \sin \lambda}\left\{2\int_0^a \frac{\mathrm{J}}{r^2}\,dr - \frac{d\mathrm{J}'}{dr}\int_0^{\pi/2} \sin\theta\, d\theta\right\}$$

$$= \frac{3\mathrm{V}a}{\pi\lambda^2 \sin\lambda}\left\{2\int_0^\lambda \frac{\mathrm{J}}{y^2}\,dy - \frac{d\mathrm{J}'}{dy}\right\}$$

where

$$y \equiv \frac{\lambda r}{a}.$$

Now

$$\int \frac{\mathrm{J}}{y^2}\,dy = -\frac{\mathrm{J}}{y} + \int \frac{1}{y}\frac{d\mathrm{J}}{dy}\,dy = -\frac{\mathrm{J}}{y} + \int\left(\frac{\sin y}{y} - \frac{\mathrm{J}}{y^2}\right)dy,$$

therefore,

$$2\int_0^\lambda \frac{\mathrm{J}}{y^2}\,dy = -\left[\frac{\mathrm{J}}{y}\right]_0^\lambda + \int_0^\lambda \frac{\sin y}{y}\,dy.$$

Also, J (y) is of the order y^2 when y is small, therefore,

$$2\int_0^\lambda \frac{\mathrm{J}}{y^2}\,dy = -\frac{\mathrm{J}'}{\lambda} + \mathrm{S}i\lambda,$$

and

$$\mu = \frac{3\mathrm{V}a}{\pi\lambda^2 \sin\lambda}(\mathrm{S}i\lambda - \sin\lambda).$$

If we replace V as a constant of the motion by μ,

$$\psi = \frac{\pi\mu a}{\lambda(\mathrm{S}i\lambda - \sin\lambda)}(\mathrm{J} - x^2\mathrm{J}')\sin^2\theta.$$

Before discussing the value of ν it will be well to get some general idea of the nature of the motions. One of the most striking peculiarities of these aggregates is the quasi-periodicity of type as λ increases from 0 to infinity. The best way to illustrate this is to use a graphical construction. Now

$$\psi \propto \{J(\lambda x) - x^2 J(\lambda)\}.$$

In fig. 2, Plate 1, the curve $y = J(\lambda)$ is drawn. P_1 corresponds to a given type (λ) of aggregate. A parabola is drawn with vertex at O and passing through P_1. Represent any abscissa to the left of λ (or of P_1) by λx, where $x < 1$. Then the differences of ordinates between the curve and the parabola up to P represent

$$J(\lambda x) - x^2 J(\lambda).$$

It is clear from the figure that, in the position P_1, this function never vanishes for $x < 1$. In the second position, P_2, however, the parabola intersects the curve at another point p. For this point (suppose $x = x_0$) ψ vanishes for all values of θ, and the corresponding current sheet is a sphere internal to the boundary. The aggregate consists of two portions with independent motions. The primary circulations are in opposite directions, and there will be *two* equatorial axes. So, as P moves on along the curve, *i.e.*, as λ increases, we get families of aggregates with three, four, &c., layers, and a corresponding number of equatorial axes. We shall denote any transition value of λ by λ_2. Each layer will have its own secondary circulation, given by the circulation round the double circuit formed by its equatorial axis, and an equator on its boundary.

Now the secondary spin velocity is given by

$$v\rho \sin\phi = \frac{\lambda}{2\pi a}\psi.$$

And since $\psi = 0$ on the boundary, it follows that

$$\nu_n = 2\pi\rho v \sin\phi, \text{ along the equatorial axis only, } = \frac{\lambda}{a}\psi_n,$$

where ψ_n is the value of ψ at the nth equatorial axis, or

$$\nu_n = \frac{\pi\mu}{Si\lambda - \sin\lambda}\{J_n - x_n^2 J'\},$$

where $J_n \equiv J(\lambda x_n)$ and $J' \equiv J(\lambda)$.

18. The moment of angular momentum is

$$\begin{aligned} \mathrm{M} &= 2\int_0^a\!\!\int_0^{\pi/2} 2\pi\rho r\,dr\,d\theta v\rho\sin\phi \\ &= \frac{2\lambda}{a}\int_0^a\!\!\int_0^{\pi/2}\psi r^2\sin\theta\,dr\,d\theta \\ &= \frac{2\pi\mu a^3}{Si\lambda-\sin\lambda}\int_0^1(\mathrm{J}-x^2\,\mathrm{J}')\,x^2\,dx\int_0^{\pi/2}\sin^3\theta\,d\theta \\ &= \tfrac{4}{3}\,\pi a^3\frac{\mu}{Si\lambda-\sin\lambda}\int_0^1\left(\frac{x}{\lambda}\sin\lambda x - x^2\cos\lambda x - x^4\mathrm{J}'\right)dx \\ &= m\frac{\mu}{Si\lambda-\sin\lambda}\left\{\left(\tfrac{1}{5}-\frac{3}{\lambda^2}\right)\left(\cos\lambda-\frac{\sin\lambda}{\lambda}\right)-\frac{\sin\lambda}{\lambda}\right\} \quad . \quad . \quad . \quad . \quad (18) \end{aligned}$$

where m denotes the volume of the aggregate.

19. The internal energy of the aggregate, supposed without translation is

$$\begin{aligned} \mathrm{E} &= \tfrac{1}{2}\iint 2\pi\rho\,d\rho\,dz\,(v^2\cos^2\phi + v^2\sin^2\phi) \\ &= \tfrac{1}{2}\iint\frac{1}{2\pi\rho}\left\{\left(\frac{d\psi}{d\rho}\right)^2+\left(\frac{d\psi}{dz}\right)^2+\frac{\lambda^2}{a^2}\psi^2\right\}d\rho\,dz \end{aligned}$$

where

$$\psi = \mathrm{A}\,(\mathrm{J}-x^2\mathrm{J}')\sin^2\theta$$

and

$$\mathrm{A} = \frac{\pi\mu a}{Si\,\lambda-\sin\lambda}.$$

Hence, as in the usual way,

$$\mathrm{E} = -\frac{1}{4\pi}\int\frac{\psi}{\rho}\frac{d\psi}{dn}\,ds - \frac{1}{4\pi}\iint\psi\left\{\frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d\psi}{d\rho}\right)+\frac{d}{dz}\left(\frac{1}{\rho}\frac{d\psi}{dz}\right)-\frac{\lambda^2}{a^2}\psi\right\}d\rho\,dz.$$

Now along the boundary $\psi = 0$. Also

$$\frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d\psi}{d\rho}\right)+\frac{d}{dz}\left(\frac{1}{\rho}\frac{d\psi}{dz}\right) = -\frac{\lambda^2}{a^2}\,\mathrm{A}\,\frac{\mathrm{J}\sin^2\theta}{\rho} = -\frac{\lambda^2\psi}{a^2\rho}-\frac{\lambda^2\mathrm{A}}{a^4}\rho\mathrm{J}',$$

therefore

$$\begin{aligned} \mathrm{E} &= \frac{\lambda^2}{2\pi a^2}\iint\frac{\psi^2 r\,dr\,d\theta}{\rho}+\frac{\lambda^2\mathrm{AJ}'}{4\pi a^4}\iint\psi\rho r\,dr\,d\theta \\ &= \frac{\lambda^2\mathrm{A}^2}{2\pi a}\left\{2\int_0^1\!\!\int_0^{\pi/2}(\mathrm{J}-x^2\mathrm{J}')^2\sin^3\theta\,d\theta\,dx + \mathrm{J}'\int_0^1\!\!\int_0^{\pi/2}(\mathrm{J}-x^2\mathrm{J}')\,x^2\sin^3\theta\,d\theta\,dx\right\} \\ &= \frac{\lambda^2\mathrm{A}^2}{3\pi a}\int_0^1\left\{x^4\mathrm{J}'^2-3\mathrm{J}'\left(\frac{x\sin\lambda x}{\lambda}-x^2\cos\lambda x\right)+\frac{2\sin^2\lambda x}{\lambda^2x^2}-2\frac{\sin 2\lambda x}{\lambda x}+2\cos^2\lambda x\right\}dx \\ &= \frac{\lambda^2\mathrm{A}^2}{3\pi a}\left\{\tfrac{1}{5}\,\mathrm{J}'^2-3\mathrm{J}'\left(\frac{3\sin\lambda}{\lambda^3}-\frac{3\cos\lambda}{\lambda^2}-\frac{\sin\lambda}{\lambda}\right)+1+\frac{\sin 2\lambda}{2\lambda}-\frac{2\sin^2\lambda}{\lambda^2}\right\} \\ &= \frac{\lambda^2\mathrm{A}^2}{3\pi a}\left\{\left(\tfrac{1}{5}-\frac{9}{\lambda^2}\right)\mathrm{J}'^2+\frac{\sin^2\lambda}{\lambda^2}-\frac{2\sin\lambda\cos\lambda}{\lambda}+1\right\} \\ &= \frac{\lambda^2\mathrm{A}^2}{3\pi a}\left\{\left(\tfrac{6}{5}-\frac{9}{\lambda^2}\right)\mathrm{J}'^2+\sin^2\lambda\right\}, \end{aligned}$$

or

$$\mathrm{E} = \frac{\pi\mu^2 a}{3} \frac{\left(\tfrac{6}{5} - \dfrac{9}{\lambda^2}\right)\mathrm{J}'^2 + \sin^2\lambda}{(\mathrm{S}i\lambda - \sin\lambda)^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (19).$$

The energy due to translation is that due to the bodily translation of the sphere $+\ \tfrac{1}{2}$ the same.

The velocity of translation is

$$\mathrm{U} = \frac{\mu}{a} \frac{\mathrm{J}' - \tfrac{1}{3}\lambda\sin\lambda}{\lambda(\mathrm{S}i\lambda - \sin\lambda)}.$$

Hence this part of the energy is

$$\tfrac{3}{2} \cdot \tfrac{1}{2} \cdot \tfrac{4}{3}\pi a^3 \mathrm{U}^2.$$

Therefore total energy is

$$= \frac{\pi\mu^2 a}{(\mathrm{S}i\lambda - \sin\lambda)^2}\left\{\left(\tfrac{2}{5} - \frac{3}{\lambda^2}\right)\mathrm{J}'^2 + \tfrac{1}{3}\sin^2\lambda + \left(\frac{1}{\lambda}\mathrm{J}' - \tfrac{1}{3}\sin\lambda\right)^2\right\}$$

$$= \frac{\pi\mu^2 a}{(\mathrm{S}i\lambda - \sin\lambda)^2}\left\{2\left(\tfrac{1}{5} - \frac{1}{\lambda^2}\right)\mathrm{J}'^2 - \frac{2}{3\lambda}\mathrm{J}'\sin\lambda + \tfrac{4}{9}\sin^2\lambda\right\} \quad . \; . \; . \; . \quad (20).$$

A verification is afforded by putting $\lambda = 0$ (Hill's vortex). Then

$$(\mathrm{S}i\lambda - \sin\lambda)^2 = \tfrac{1}{81}\lambda^6.$$

$$\text{Large bracket} = \tfrac{2}{5\,.\,7\,.\,9\,.\,9}\lambda^6.$$

$$\mathrm{E} = \tfrac{2}{35}\pi\mu^2 a,$$

which is correct.

The preceding formulæ refer to the whole aggregate. When, however, $\lambda >$ the lowest λ_2, there are more than one component, and it will be well to give the requisite formulæ for each of these separately. Denote $\lambda r_n/a$ by y_n, where r_n is the radius of the nth interface from the centre. Also for shortness let $\mathrm{S}(x)$ denote the function $\mathrm{S}ix - \sin x$. Then

$$\frac{\mu_n}{\mu} = \frac{\mathrm{S}(y_n) - \mathrm{S}(y_{n-1})}{\mathrm{S}(\lambda)} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (21).$$

$$\frac{\nu_n}{\mu_n} = \frac{\pi}{\mathrm{S}(y_n) - \mathrm{S}(y_{n-1})}\{\mathrm{J}_n - x_n^2\,\mathrm{J}'\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (22),$$

J_n denoting the value of J at the equatorial axis.

$$\mathrm{M}_n = \frac{m\mu}{\lambda^3(\mathrm{S}i\lambda - \sin\lambda)}\int_{y_{n-1}}^{y_n}\left(y\sin y - y^2\cos y - y^4\frac{\mathrm{J}'}{\lambda^2}\right)dy$$

$$= \frac{1}{\lambda^3}\frac{m\mu_n}{\mathrm{S}(y_n) - \mathrm{S}(y_{n-1})}\left\{3y_n\mathrm{J}(y_n) - 3y_{n-1}\mathrm{J}(y_{n-1}) - y_n^2\sin y_n + y_{n-1}^2\sin y_{n-1}\right.$$

$$\left. - \frac{y_n^5 - y_{n-1}^5}{5} \cdot \frac{\mathrm{J}(\lambda)}{\lambda^2}\right\}.$$

But

$$Jy - y^2 \frac{J\lambda}{\lambda^2} = 0.$$

Hence

$$M_n = \frac{1}{\lambda^3} \frac{m\mu_n}{S(y_n) - S(y_{n-1})} \left[\left\{ 3(y_n^3 - y_{n-1}^3) - \frac{y_n^5 - y_{n-1}^5}{5} \right\} \frac{J(\lambda)}{\lambda^2} - y_n^2 \sin y_n + y_{n-1}^2 \sin y_{n-1} \right]. \quad (23).$$

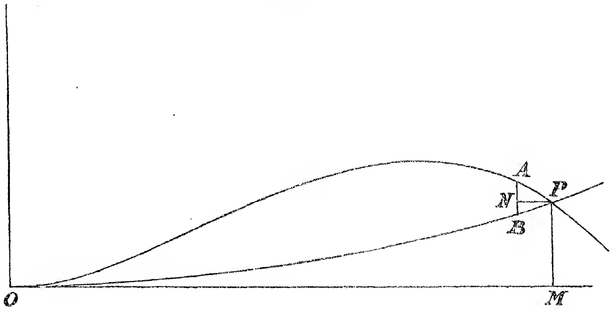
20. The velocity of translation is given by

$$U = -\frac{\mu}{3a} \frac{\lambda \sin\lambda - 3(\sin\lambda/\lambda - \cos\lambda)}{\lambda(Si\lambda - \sin\lambda)}$$

$$= -\frac{\mu}{3a(Si\lambda - \sin\lambda)} \frac{d}{d(\lambda x)} \{J(\lambda x) - x^2 J(\lambda)\}_{x=1}.$$

To see how this varies with the parameter λ, refer to the graphical construction in fig. 2, Plate 1. The curve J and the parabola intersect in P. If A be a point on

Fig. 3.



the curve (fig. 3), and B on the parabola with the same abscissa near P, and PN be the perpendicular on AB,

$$U = \frac{\mu}{3a(Si\lambda - \sin\lambda)} \frac{AB}{PN}$$

$$= -\frac{\mu}{3a(Si\lambda - \sin\lambda)} \frac{\sin(\alpha - \beta)}{\cos\alpha \cos\beta}$$

$$= -\frac{\mu}{3aA} \cdot \frac{\sin(\alpha - \beta)}{\cos\alpha \cos\beta}.$$

Where α, β are the angles which the tangents to the curve and the parabola at P make with the axis of x, and A denotes the area of the curve OAPMO.

The factor $\frac{\mu}{3a(Si\lambda - \sin\lambda)}$ is always finite, except for $\lambda = 0$, and positive. It is then easy to see in general how the velocity alters as the parameter λ increases.

As P (fig. 2, Plate 1), travels along the curve, U is positive. Leaving out of sight for the present its value for λ small, it later on diminishes to zero when P reaches a certain point Q where the parabola touches the curve. It then changes sign and remains negative until P reaches another Q point where the parabola again touches the curve, and so on.

We shall call the values of λ corresponding to the Q points the λ_2 values, and denote them in order by $\lambda_2^{(1)}, \lambda_2^{(2)}, \ldots \lambda_2^{(n)}$. Thus for values of $\lambda < \lambda_2^{(1)}$ the aggregate moves in the direction of the rotational flow up the axis. At $\lambda = \lambda_2^{(1)}$ the aggregate is at rest, the velocity of the fluid on the boundary is zero; as λ increases beyond this, the aggregate takes on another layer with primary rotation in the opposite direction, and it moves in the fluid in a direction opposed to the rotational motion of the innermost layer. It regredes relatively to this. The velocity at first increases and then diminishes until P reaches the second λ_2 point, when the corresponding aggregate is at rest in the fluid, and so on.

The periodic nature of the aggregates is thus evident. We get for example a whole periodic family of aggregates whose peculiar property is that they remain at rest in the fluid. The members of the family differ, amongst other things, in the number of independent layers each possesses.

So we get another family formed by values of λ, corresponding to points where the J-curve cuts the axis of x. We will call values of λ, corresponding to these the λ_1 parameters, and denote the orders in the same way as for the λ_2 parameters. As we shall see shortly, the distinguishing property of this family is that in each of them the vortex lines and the stream lines coincide.

For small values of λ it is preferable to express the value of U in terms of the lowest powers of λ.

It is easy to show that

$$\mathrm{S}i\lambda - \sin\lambda = \frac{2}{3.3\,!}\lambda^3 - \ldots - (-)^n \frac{2n}{2n+1}\frac{\lambda^{2n+1}}{(2n+1)\,!},$$

whence

$$\mathrm{U} = \frac{\mu}{5a}\left(1 - \tfrac{2}{175}\lambda^2\right).$$

This gives for $\lambda = 0$ the value of U already known for HILL's vortex.

The curve $y = \mathrm{U}/\mathrm{U}_0$, where U_0 is the velocity of the non-gyrostatic aggregate of same cyclic constant and volume, is drawn in fig. 3, Plate 1, up to $\lambda_2^{(6)}$. The periodic quality is evident.

21. The directions of the lines of flow and of the vortex lines are given by

$$\tan\phi = \frac{v\rho\sin\phi}{v\rho\cos\phi} = \frac{\lambda}{a}\frac{\psi}{\frac{d\psi}{dn}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24),$$

$$\tan\chi = \frac{\omega\rho\sin\chi}{\omega\rho\cos\chi} = -\frac{\lambda}{a}\frac{\mathrm{J}\sin^2\theta}{\frac{d}{dn}\left\{(\mathrm{J} - x^2\mathrm{J}')\sin^2\theta\right\}}.$$

Hence

$$\tan\chi = -\tan\phi - \frac{\lambda}{a}\,\frac{J'x^2\sin\,\theta}{\dfrac{d}{dn}\left\{(J - x^2J')\sin^2\theta\right\}} \quad . \quad . \quad . \quad . \quad (25),$$

also

$$\frac{\tan\phi}{\tan\chi} = -1 + x^2\frac{J'}{J}.$$

Equation (25) shows that when $J' = 0$, *i.e.*, for the λ_1 parameters, the stream lines and vortex lines coincide. (It is to be remembered that we have supposed in the foregoing that ϕ and χ lie on opposite sides of meridian lines, and therefore $\tan\phi = -\tan\chi$ means that they lie on the same side and coincide.)

I. From $\lambda = 0$ up to $\lambda = \lambda_1^{(1)}$, $J > x^2J'$ and $J - x^2J' < J$. Hence between these limits, the stream lines and vortex line are on the same side of the meridians, and $\chi > \phi$, *i.e.*, the stream lines lie between the vortex lines and meridians. At $\lambda = \lambda_1^{(1)}$ they coincide.

II. Between $\lambda_1^{(1)}$ and $\lambda_2^{(1)}$ $J > x^2J'$, but $J - x^2J' > J$. For any given λ, J changes from + to − as x passes through the value $\lambda x = \lambda_1^{(1)}$. For this value of x, or $r = \frac{\lambda_1}{\lambda}a$, $\chi = 0$. Thus, for an aggregate whose parameter λ lies between the first λ_1 and λ_2 roots, the vortex lines lie between the stream lines and the meridians for all points at a less distance from the centre than $r = \frac{\lambda_1}{\lambda}a$. At this distance $\chi = 0$, or the vortex lines coincide with the meridian planes, and beyond this distance up to the boundary the vortex lines and stream lines are on opposite sides of the meridians.

For values of λ between the first and second λ_2 parameters we have to deal with two layers. In the outer $J - x^2J'$ is negative, whilst J is negative between $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$, positive between $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$. Referring to fig. 2, Plate 1, let the point p where the parabola cuts the J curve be given by λ', corresponding in the aggregate to a distance from the centre $\lambda x = \lambda'$ or $r = \frac{\lambda'}{\lambda}a$. It is clear that J (λ) and J (λ') are of the same sign. Hence, if λ lies between $\lambda_2^{(1)}$ and $\lambda_1^{(2)}$ (corresponding to P between Q_1 and R_2), λ' lies between $\lambda_1^{(1)}$ and $\lambda_2^{(1)}$, whereas if λ lies between $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$, λ' lies between 0 and $\lambda_1^{(1)}$—or, taking closer limits still, between π and $\lambda_1^{(1)}$. We find, therefore, the following results.

III. P between Q_1 and R_2. In the inner spherical nucleus the vortex lines lie on the same side of the stream lines as the meridians—they are, in fact, exactly similar to the second category. At the boundary between the central nucleus and the outer layer $\phi = 0$, the stream lines coincide with the meridians. In the outer layer the stream lines lie on the other side of the meridian, with the vortex lines beyond. When P coincides with R_2 or λ is the second λ_1 parameter the stream lines coincide with the vortex lines again, but on the opposite side of the meridians.

IV. For P between R_2 and Q_2, we get still two layers, the boundary being given by (say) λ' (P at p), where $J\lambda'$ and $J\lambda$ are both positive. $J - x^2J'$ is positive between 0 and $x = \lambda'/\lambda$ and negative between $x = \lambda'/\lambda$ and 1. In the inner spherical nucleus ($r = 0$ to $r = \lambda' a/\lambda$) the stream lines lie between the vortex lines and the meridians (similar to the first category). At the interface the stream-lines coincide with the meridian. In the outer layer the stream lines and vortex lines lie on opposite sides of the meridian for points whose distance from the centre are less than $\frac{\lambda_1^{(1)}}{\lambda} a$, or greater than $\frac{\lambda_1^{(2)}}{\lambda} a$. For points at a distance $\frac{\lambda'_1}{\lambda} a$ and $\frac{\lambda_1^{(2)}}{\lambda} a$, the vortex lines coincide with the meridians, and between them the two lines lie on the same side of the meridian. In the same way the behaviour for aggregates whose parameter is greater than $\lambda_2^{(2)}$, may be determined. The periodic nature of the aggregate is again very clearly seen.

It is perhaps easier to describe the nature of the changes above indicated by supposing our eyes placed in a prolongation of the polar axis. Call the vortex lines blue lines and the stream lines red lines, and suppose for λ small that the stream or red lines lie on the right of the meridians. For $\lambda = 0$, or HILL's vortex, the red lines lie along meridians and the blue lines perpendicular to these, along parallels of latitude. As λ increases the red and blue lines swing round towards each other, the reds to the right and the blues to the left, and this goes on with increasing values of λ up to $\lambda_1^{(1)}$, when they coincide. Beyond $\lambda = \lambda_1^{(1)}$ and up to $\lambda = \lambda_2^{(1)}$ the red and blue lines interchange their relative positions. In any given aggregate the blue lines move more and more towards meridians as we pass from the centre outwards. At a distance $\frac{\lambda_1^{(1)}}{\lambda} a$ from the centre the blue lines all coincide with the meridians, both red and blue lines are swinging round to the left. Beyond the distance $\frac{\lambda_1^{(1)}}{\lambda} a$ the blue lines cross to the left of the meridians and the red lines close up towards the meridians until at the surface of the aggregate they coincide with them.

Between $\lambda_2^{(1)}$ and $\lambda_2^{(2)}$ we have doublets. The aggregates lying between $\lambda_2^{(1)}$ and $\lambda_1^{(2)}$ and between $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$ are however essentially different.

In the first set in the central nucleus the blue lines lie to the left of the red, and both to the right of the meridians for points near the centre. As we pass outwards from the centre they swing round to the left, the blue lines swing past the meridians whilst at the surface of the nucleus the red lines just reach it. Beyond, in the outer layer as we pass out, the blue and red swing further to the left, and later at least the red swing back again towards the meridian, coinciding with it at the surface. When $\lambda = \lambda_1^{(2)}$ red and blue coincide everywhere. They lie to the right in the inner nucleus and to the left in the outer layer.

Between $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$ we get aggregates in which red and blue lines again change sides. In the inner nucleus both lie to the right of the meridian, blue furthest out. They close up to the meridian as we pass out from the centre to the nucleus surface. In the outer layer the red lines swing further to the left and back again, the blue lines follow after in the same way, crossing the meridian twice; once in each direction.

Beyond $\lambda_2^{(2)}$ we get triplets.

In general, between $\lambda_1^{(n)}$ and $\lambda_1^{(n+1)}$ the blue lines lie to the right of the red or the opposite according as n is even or odd. They coincide for the λ_1 parameter. Also, if n is even, both lie to the right of the meridian for the inner nucleus, the reds to the left for the second layer, to the right for the third, and so on. Whilst the opposite takes place if n is odd.

The forms of the spirals may be obtained by finding the polar equations to their projections on the equatorial plane. Let (ρ, η) be the polar co-ordinates of a point on the projection of a flow; (ρ, ς) of a vortex line lying on a given sheet ψ. Then

$$\rho \frac{d\eta}{ds} = \tan \phi, \qquad \rho \frac{d\varsigma}{ds} = \tan \chi,$$

where ds is an element of a meridian curve. Hence

$$d\eta = \frac{\lambda}{a} \frac{\psi \, ds}{\rho \dfrac{d\psi}{dn}} = \frac{\lambda}{a} \frac{\psi \, dr}{\rho \dfrac{d\psi}{r \, d\theta}}.$$

Provided dr is not perpendicular to ds, *i.e.*, on the outer boundary, but then $\psi = 0$ and $\eta = 0$.

$$d\eta = \frac{\lambda}{a} \frac{dr}{2 \cos \theta}, \qquad \eta = \frac{\lambda}{2} \int_{x_1}^{x} \frac{dx}{\cos \theta},$$

where x_1 corresponds to the inner circle of the two in which the current sheet ψ cuts the equatorial plane. The total angular pitch of the spiral is

$$\lambda \int_{x_1}^{x_2} \frac{dx}{\cos \theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26),$$

where x_1, x_2 are the two roots of

$$\mathrm{J}(\lambda x) - x^2 \mathrm{J}\lambda = \frac{\lambda \psi (Si\lambda - \sin \lambda)}{\pi \mu a} = b, \text{ say.}$$

The above may also be written

$$\eta = \tfrac{1}{2}\lambda \int_{x_1}^{x} \left\{ \frac{\mathrm{J} - x^2 \mathrm{J}'}{\mathrm{J} - x^2 \mathrm{J}' - b} \right\}^{\frac{1}{2}} dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

Equation (26) enables us easily to determine the form graphically when the surfaces ψ are drawn. So

$$\varsigma = \eta + \frac{\lambda}{2} \mathrm{J}' \int_{x_1}^{x} \{ (\mathrm{J} - x^2 \mathrm{J}')(\mathrm{J} - x^2 \mathrm{J}' - b) \}^{-\frac{1}{2}} x^2 \, dx \quad . \quad . \quad . \quad (28),$$

the case of a spherical boundary being excepted as before.

For the outside stream-lines the pitch is

$$\eta = 2 \times \frac{\lambda}{2} \int_0^1 dx = \lambda.$$

For values of λ, however, lying beyond $\lambda_2^{(1)}$ there are several layers in which the stream-lines are distinct. If $x_1, x_2, x_3 \ldots$ denote the values of x corresponding to the interfaces of the layers, the pitches of the stream-lines on those surfaces as we pass outwards are

$$(x_1 - 0)\,\lambda, \qquad (x_2 - x_1)\,\lambda, \qquad (x_3 - x_2)\,\lambda, \text{ \&c.}$$

We have seen that on these surfaces the stream-lines coincide with the meridian. These parts therefore produce no part of the pitch. The twist must be supposed as taking place in the part of the stream-line along the polar axes. It is easy to see that this is so by considering current sheets near the interfaces.

We may therefore regard the physical meaning of λ to be the criterion of the total external pitch of the stream-lines. We will return to the consideration of the pitch, and the shape of these lines later.

The total angular pitch of a stream spiral on any stream sheet ψ can easily be expressed in terms of the volume of the fluid inside that sheet. For

$$\begin{aligned} d\varsigma &= d\eta - \frac{ds}{\rho} \cdot \frac{\lambda J'}{a^3} \cdot \frac{\rho^2}{\dfrac{d\psi}{dn}} \times \frac{\pi\mu a}{\lambda(\mathrm{Si}\lambda - \sin\lambda)} \\ &= d\eta - \frac{\lambda J'}{2\pi a^3} \frac{2\pi\rho\, ds\, dn}{d\psi'} \text{ if } \psi = \frac{\pi\mu a}{\lambda(\mathrm{Si}\lambda - \sin\lambda)}\psi'. \end{aligned}$$

Integrate round the stream surface

$$\varsigma = \eta - \frac{\lambda J'}{2\pi a^3} \frac{d}{d\psi'} \int 2\pi\rho\, ds\, dn = \eta - \frac{\lambda J'}{2\pi a^3} \frac{dm}{d\psi'} \quad . \quad . \quad . \quad . \quad . \quad (29),$$

where m denotes the volume inside ψ.

22. The discriminating properties of the λ_1 and λ_2 parameters make it important to determine their values. The λ_1 parameters are the roots of the equation

$$J(\lambda) \equiv \frac{\sin\lambda}{\lambda} - \cos\lambda = 0, \quad \text{or} \quad \tan\lambda = \lambda$$

The large roots are clearly nearly $(2n + 1)\frac{\pi}{2}$.

Put

$$\lambda = (2n + 1)\frac{\pi}{2} - y = \alpha - y \text{ say.}$$

Then

$$\frac{\cos y}{a - y} - \sin y = 0.$$

Expanding this in powers of y, it is easily proved by successive approximation that

$$y = \frac{1}{a} + \frac{2}{3a^3} + \frac{13}{15a^5}$$

or

$$\lambda_1^{(n)} = (2n+1)\frac{\pi}{2} - \frac{2}{(2n+1)\pi} - \frac{16}{3(2n+1)^3\pi^3} - \frac{13 \times 32}{15(2n+1)^5\pi^5}$$

$$= 1{\cdot}57079\,(2n+1) - \frac{{\cdot}63662}{2n+1} - \frac{{\cdot}17201}{(2n+1)^3} - \frac{{\cdot}03558}{(2n+1)^5} \quad . \quad . \quad . \quad (30).$$

The first root is by numerical calculation

$$\lambda = 4{\cdot}49341 = 257^\circ\, 27'\, 10''$$

The foregoing formula gives for this case ($n = 1$)

$$\lambda = 4{\cdot}49366.$$

For higher values the formula is correct to five places at least
The first three roots are

$$\left.\begin{aligned} 4{\cdot}49341 &= 270^\circ - 12^\circ\, 32'\, 50' \\ 7{\cdot}72528 &= 450^\circ - 7^\circ\, 22'\, 27'' \\ 10{\cdot}90408 &= 630^\circ - 5^\circ\, 14'\, 23'' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

The λ_2 parameters are roots of the equation

$$\cot\lambda = \frac{1}{\lambda} - \frac{\lambda}{3}\,.$$

The large roots are clearly nearly $n\pi = n\pi - y$ say, where

$$\cot y = \frac{n\pi - y}{3} - \frac{1}{n\pi - y}\,,$$

or

$$\cos y = \left(\frac{n\pi - y}{3} - \frac{1}{n\pi - y}\right)\sin y.$$

Writing $n\pi \equiv \beta$, and expanding in terms of y it is easy to prove, as in the former case, that

$$y = \frac{3}{\beta} + \tfrac{1}{3}\left(\frac{3}{\beta}\right)^3 + \tfrac{1}{5}\left(\frac{3}{\beta}\right)^5$$

$$\lambda = n\pi - \frac{3}{n\pi} - \tfrac{1}{3}\cdot\left(\frac{3}{n\pi}\right)^3 - \tfrac{1}{5}\left(\frac{3}{n\pi}\right)^5$$

$$= 3{\cdot}14159n - \frac{{\cdot}95493}{n} - \frac{{\cdot}29026}{n^3} - \frac{{\cdot}15881}{n^5} \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$

There is no root corresponding to $n = 1$. The first root is

$$\lambda_2 = 5{\cdot}76346 = 360^\circ - 29^\circ\, 46'\, 41''.$$

The formula gives for this root

$$\lambda = 5{\cdot}76448.$$

For $n > 2$ it is exact to five places.

The first three roots are

$$\left.\begin{aligned} 5{\cdot}76346 &= 360^\circ - 29^\circ\, 46'\, 41'' \\ 9{\cdot}09506 &= 540^\circ - 18^\circ\, 53'\, 29'' \\ 12{\cdot}32296 &= 720^\circ - 13^\circ\, 56'\, 48'' \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (33).$$

23. *Equatorial Axes.*—An equatorial axis is the line of particles which remains at rest. It is given by the equation

$$\frac{d\psi}{dr} = 0, \qquad \text{when } \theta = 0,$$

or by

$$\frac{dJ}{dx} - \frac{d}{dx}(x^2 J') = 0.$$

The positions of the axes are, therefore, readily observed by means of the graphical construction in fig. 2, Plate 1. They depend on the abscissæ of points for which the tangents to the J curve and the parabola are parallel. For values of $\lambda > \lambda_1^{(1)}$, the inclination of the parabola to the axis of x is always small. Hence the equatorial axes must always be near the crests (or bottoms) of the J curve, *i.e.*, near values $(2m + 1)\frac{1}{2}\pi$.

The equation for the axes becomes, if y be put for λx,

$$\cos y + \left(y - \frac{1}{y}\right)\sin y - 2y^2 \frac{J'}{\lambda^2} = 0 \quad \ldots\ldots\ldots \quad (34),$$

in which the roots $< \lambda$ are required.

As the values of the secondary cyclic constants and other important properties depend on the position of the equatorial axes, it will be necessary to determine their values. We shall do this (1) for the case of λ small, and (2) for the case of λ large. As, however, the case of the λ_1 values is special, we shall treat these separately. In the case of values other than λ_1, say, *e.g.*, λ_2 parameters, all the axes of any aggregate depend on the particular λ value. In the case of λ_1, however, they are independent of the particular λ_1. In fact, the successive λ_1 aggregates may be built up by taking any one and putting outside of this a suitable vortex shell. Moreover, the values of the axes for the λ_1 roots are the crests, and bottoms, of the J curve, and so are important for their own sakes.

Case of λ *small.*—Here y is also small. If equation (34) be expanded in powers of y and λ, there results

$$2\lambda^2 y^2 \Sigma_1 (-)^n \frac{\lambda^{2n-2}}{(2n+3)(2n+1)!} + 4\Sigma_2 (-)^n \frac{n^2}{(2n+1)!} y^{2n} = 0.$$

Dividing by $2y^2/15$, this may be written

$$y^2 = \frac{\lambda^2}{2} + 30\Sigma_2 (-)^n \frac{n+1}{(2n+3)!} \{(n+1) y^{2n} - \lambda^{2n}\},$$

whence y can be expressed in terms of λ by successive approximation. To λ^6 it will be found that

$$y^2 = \frac{\lambda^2}{2} \left\{1 - \frac{\lambda^2}{112} - \tfrac{50}{27} \left(\frac{\lambda^2}{112}\right)^2 \right\},$$

$$y = \frac{\lambda}{\sqrt{2}} \left\{1 - \frac{\lambda^2}{224} - \tfrac{227}{54} \left(\frac{\lambda^2}{224}\right)^2 \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35).$$

This gives the equatorial axis at

$$r = \frac{a}{\sqrt{2}} \left\{1 - \frac{\lambda^2}{224} - \tfrac{227}{54} \left(\frac{\lambda^2}{224}\right)^2 \right\}.$$

When $\lambda = 0$, this agrees with HILL's vortex.

Case of λ_1.—The equation in y for this case becomes

$$\cos y + \left(y - \frac{1}{y}\right) \sin y = 0,$$

y is always nearly $n\pi = n\pi - z$ say, where z is small. Then

$$\cos z - \left(n\pi - z - \frac{1}{n\pi - z}\right) \sin z = 0.$$

Whence

$$y = n\pi - \frac{1}{n\pi} - \frac{5}{3(n\pi)^3} - \frac{7^3}{15(n\pi)^5}$$

$$= n\pi - \frac{\cdot 31831}{n} - \frac{\cdot 05375}{n^3} - \frac{\cdot 01590}{n^5}.$$

This formula gives for the two first roots

2·75363, 6·11682.

The values obtained by numerical calculation are

2·74371, 6·11676.

The roots beyond this are therefore given by the formula correct to five places. The radii of the equatorial axes are $r = ya/\lambda$. Hence using the values of λ_1 given in (31), the first three are. For $\lambda_1^{(1)}$,

$$r = \frac{2{\cdot}74371}{4{\cdot}49341}\,a = {\cdot}61062\,a.$$

For $\lambda_1^{(2)}$,

$$\left.\begin{aligned} r_2 &= \frac{6{\cdot}11676}{7{\cdot}72528}\,a = {\cdot}79179\,a \\ r_1 &= \frac{2{\cdot}74371}{7{\cdot}72528}\,a = {\cdot}35516\,a \end{aligned}\right\}.$$

For $\lambda_1^{(3)}$

$$\left.\begin{aligned} r_3 &= \frac{9{\cdot}31663}{10{\cdot}90408}\,a = {\cdot}85442\,a \\ r_2 &= \frac{6{\cdot}11676}{10{\cdot}90408}\,a = {\cdot}56096\,a \\ r_1 &= \frac{2{\cdot}74371}{10{\cdot}90408}\,a = {\cdot}25162\,a \end{aligned}\right\}.$$

Case of λ large.—The number of equatorial axes depends on the order of the λ_2 parameter next greater than λ. If λ lie between $\lambda_2^{(n-1)}$ and $\lambda_2^{(n)}$, there are n such axes. It seems then natural to refer the magnitude of λ to $\lambda_2^{(n)}$. Suppose then

$$\lambda = \lambda_2^{(n)} - \mathrm{X},$$

where the maximum value of X is about π,—*or* we may write $\lambda = \lambda_2^{(n-1)} + \mathrm{X}$, and if both be allowed X will have a maximum of the order $\frac{1}{2}\pi$.

$$\lambda_2^{(n)} = n\pi - \frac{3}{n\pi} - \tfrac{1}{3}\left(\frac{3}{n\pi}\right)^2 - \tfrac{1}{5}\left(\frac{3}{\pi}\right)^5.$$

The equation in y is

$$\cos y + \left(y - \frac{1}{y}\right)\sin y - 2y^2\,\frac{\mathrm{J}\lambda}{\lambda^2} = 0,$$

in which the first n roots are to be determined. For small roots the parabola of fig. 3 is almost coincident with the axis of x, and consequently the small y roots are very nearly equal to the corresponding values for λ_1. It will be best to obtain an expression for the large roots and then see how far back it holds for the smaller roots. Clearly y is always near $m\pi$ where m is an integer $< n$.

Put

$$y = m\pi + z = \alpha + z \text{ say}.$$

Then

$$\frac{\cos z}{y} + \left(1 - \frac{1}{y^2}\right)\sin z - (-)^m\,2y\,\frac{\mathrm{J}\lambda}{\lambda^2} = 0.$$

J (λ) may be either + or −, it is of order of magnitude 1 at most.

Since z is not large (it is of order $1/\alpha$), we get

$$\frac{1}{\alpha}\left(1-\frac{z}{\alpha}+\frac{z^2}{\alpha^2}\right)\left(1-\frac{z^2}{2}+\frac{z^4}{4!}\right)+\left(1-\frac{1}{\alpha^2}+\frac{2z}{\alpha^3}\right)$$
$$\times\left(z-\frac{z^3}{6}+\frac{z^5}{5!}\right)-(-)^m\,2\frac{\alpha}{\lambda}\left(1+\frac{z}{\alpha}\right)\frac{\mathrm{J}\lambda}{\lambda}=0.$$

Write

$$2(-)^m\frac{\alpha}{\lambda}\cdot\frac{\mathrm{J}\lambda}{\lambda}\equiv\frac{b}{\alpha}.$$

The greatest value of α/λ is < 1. $\mathrm{J}'\lambda/\lambda$ is of order $1/\lambda$, therefore at least of order $1/\alpha$. Hence in the most unfavourable cases b is $\overline{\overline{<}}\, 2$. The above equation can be written

$$-z=\frac{1}{\alpha}-\frac{b}{\alpha}-\frac{bz}{\alpha^2}-\frac{2z}{\alpha^2}-\frac{z^2}{2\alpha}-\frac{z^3}{6}+\frac{2}{3}\frac{z^3}{\alpha^2}+\frac{3z^2}{\alpha^3}+\frac{z^4}{\alpha.4!}+\frac{z^5}{5!},$$

$$z=\frac{b}{\alpha}-\frac{1}{\alpha}=\frac{b-1}{\alpha}, \qquad \text{(1st approx.)}$$

$$z=\frac{b-1}{\alpha}+\frac{(b+2)(b-1)}{\alpha^3}+\frac{(b-1)^2}{2\alpha^3}+\frac{(b-1)^3}{6\alpha^3}, \quad \text{(2nd approx.)}$$

$$=\frac{b-1}{\alpha}+\frac{(b-1)(b+2)(b+5)}{6\alpha^3} \qquad \ldots\ldots\ldots\ldots \qquad (36).$$

It will be convenient to put $b-1\equiv c$. Then

$$z=\frac{c}{\alpha}+\frac{c(c+3)(c+6)}{6\alpha^3}+\frac{1}{\alpha^5}\left\{\frac{c(c+3)(c+6)(c^2+4c+6)}{12}-3c^2-\tfrac{2}{3}c^3-\frac{c^4}{4!}-\frac{c^5}{5!}\right\}.$$

If λ is a λ_1 root, $c=-1$ and

$$z=-\frac{1}{\alpha}-\frac{5}{3\alpha^3}-\frac{7^3}{15\alpha^5},$$

which agrees with the result already found.

24. *The Spiral Forms taken by the Lines of Flow and Vortex Filaments.*—The equations determining these are given in § (21). Unfortunately, however, they are not integrable in finite forms.

We give a graphical method for the stream-lines later. At present it is proposed to determine (1) the forms of the stream and vortex lines when λ is small, (2) the pitch of the spirals near the equatorial axes, and (3) the pitch of the same on the outer surface.

Let the stream surface ψ, the streams and filaments on which we have to investigate, cut the equatorial plane in circles given by $r/a = x_1$ and x_2. Then

$$\eta=\frac{\lambda}{2}\int_{x_1}^{x}\sqrt{\left\{\frac{\mathrm{J}-x^2\mathrm{J}'}{\mathrm{J}-x^2\mathrm{J}'-b}\right\}}\,dx$$

where

$$b = \frac{\lambda}{\pi\mu a}(\mathrm{S}i\lambda - \sin\lambda)\,\psi.$$

In determining the vortex filaments we will take ς to be measured in the same direction as η : that is, to the right of meridians as looked at from the polar axis.

In this case

$$\varsigma = \lambda\int_{x_1}^{x} \frac{\mathrm{J}\sin^2\theta\,dx}{\sin\theta\,(\mathrm{J} - x^2\mathrm{J}')\,2\sin\theta\cos\theta} = \frac{\lambda}{2}\int_{x_1}^{x} \frac{\mathrm{J}\,dx}{\sqrt{\{(\mathrm{J} - x^2\mathrm{J}')(\mathrm{J} - x^2\mathrm{J}' - b)\}}}$$

or

$$\varsigma = \eta + \frac{\lambda}{2}\mathrm{J}'\int_{x_1}^{x} \frac{x^2\,dx}{\sqrt{(\mathrm{J} - x^2\mathrm{J}')(\mathrm{J} - x^2\mathrm{J}' - b)}}\,.$$

(1.) Case of λ small.

$$\mathrm{J} - x^2\mathrm{J}' = \tfrac{1}{3}\lambda^4 x^2\left\{-\frac{x^2-1}{10} + \frac{3\lambda^2}{7\,.\,5\,!}(x^4 - 1)\right\}$$

also b is of order λ^4. Put

$$b = \tfrac{1}{30}\lambda^4 c.$$

Hence

$$\eta = \frac{\lambda}{2}\int_{x_1}^{x}\sqrt{\left\{\frac{(1-x^2)\left(1 - \dfrac{\lambda^2}{28}\,\overline{x^2+1}\right)}{x^2(1-x^2) - \dfrac{\lambda^2 x^2}{28}(1-x^4) - c}\right\}}\;x\,dx.$$

x_1, x_2 are the roots of the denominator equated to 0, viz., of

$$x^4 - x^2 + c = \frac{\lambda^2}{28}(x^6 - x^2).$$

A first approximation is

$$x^2 = \frac{1 \pm \sqrt{1-4c}}{2} = r_1 \text{ or } r_2 \text{ (say), where } r_1 \text{ denotes the smaller root.}$$

Let for a second approximation

$$x^2 = r_1 + \xi, \text{ where } \xi \text{ is of order } \lambda^2.$$

$$x^4 - x^2 + c = r_1^2 + 2r_1\xi - r_1 - \xi + c = (2r_1 - 1)\,\xi.$$

$$x^6 - x^2 = r_1^3 + 3r_1\xi - r_1 - \xi.$$

Therefore,

$$(2r_1 - 1)\xi = \frac{\lambda^2}{28}(r_1^3 - r_1).$$

$$\xi = \frac{\lambda^2}{28}\,\frac{r_1^3 - r_1}{2r_1 - 1} = -\frac{\lambda^2}{28}\,\frac{r_1 r_2 (r_1 + 1)}{r_1 - r_2}\,.$$

Since $r_1 + r_2 = 1$, also $r_1 r_2 = c$,

$$\xi = -\frac{\lambda^2 c}{28} \cdot \frac{r_1+1}{r_1-r_2}.$$

Hence the roots are

$$x_1^2 = r_1 + \frac{\lambda^2 c}{28} \cdot \frac{r_1+1}{r_2-r_1}$$

and

$$x_2^2 = r_2 - \frac{\lambda^2 c}{28} \cdot \frac{r_2+1}{r_2-r_1},$$

and the denominator becomes

$$(x^2 - x_1^2)(x_2^2 - x^2)\left(1 - \frac{\lambda^2}{28} - \frac{\lambda^2}{28} x^2\right).$$

Whence

$$\eta = \frac{\lambda}{2} \int_{x_1}^{x} \sqrt{\frac{(1-x^2)\left(1 - \frac{\lambda^2}{28} - \frac{\lambda^2}{28} x^2\right)}{(x^2 - x_1^2)(x_2^2 - x^2)\left(1 - \frac{\lambda^2}{28} - \frac{\lambda^2}{28} x^2\right)}}\, x\, dx,$$

$$= \frac{\lambda}{4} \int_{y_1}^{y} \sqrt{\left\{\frac{1-y}{(y-y_1)(y_2-y)}\right\}}\, dy, \text{ where } y = x^2.$$

In this put

$$y = \frac{y_2+y_1}{2} - \frac{y_2-y_1}{2} \cos\theta.$$

So that

$$y - y_1 = \frac{y_2-y_1}{2}(1 - \cos\theta) \qquad y_2 - y = \frac{y_2-y_1}{2}(1 + \cos\theta).$$

Then

$$\eta = \frac{\lambda}{4} \int_0^{\theta} \sqrt{1-y}\, d\theta,$$

$$= \frac{\lambda}{4} \int_0^{\theta'} \sqrt{\left\{1 - y_1 - (y_2 - y_1)\sin^2\frac{\theta}{2}\right\}}\, d\theta,$$

$$= \frac{\lambda}{2} \sqrt{1 - y_1} \int_0^{\phi} \sqrt{1 - k^2 \sin^2\phi}\, d\phi,$$

$$= \frac{\lambda}{2} \sqrt{1 - y_1}\, \mathrm{E}(k, \phi), \text{ where } \sin^2\phi = \frac{x^2 - x_1^2}{x_2^2 - x_1^2} \quad . \quad . \quad . \quad . \quad . \quad (37),$$

and

$$k^2 = \frac{y_2 - y_1}{1 - y_1} = \frac{x_2^2 - x_1^2}{1 - x_1^2},$$

$$= \frac{r_2 - r_1 - \frac{\lambda^2 c}{28} \frac{r_2 + r_1 + 2}{r_2 - r_1}}{r + r_1 - r_1 - \frac{\lambda^2 c}{28} \frac{r_1 + 1}{r_2 - r_1}}$$

$$= 2 \frac{1 - 4c - \frac{3\lambda^2 c}{28}}{1 - 4c + \sqrt{1 - 4c} - \frac{\lambda^2 c}{28}\left(3 - \sqrt{1 - 4c}\right)}.$$

At the equatorial axis $k = 0$, on the surface $k = 1$. Thus k increases from 0 to 1 for the various current sheets in order from the axis to the surface. The pitch of the helix on any sheet is

$$\text{Pitch} = \lambda \sqrt{1 - y_1}\, \mathrm{E}.$$

At the surface this is λ, at the axis it is

$$= \lambda \sqrt{1 - y_0} \,.\, \frac{\pi}{2} = \frac{\pi\lambda}{2} \sqrt{\left\{1 - \tfrac{1}{2}\left(1 - \frac{\lambda^2}{112}\right)\right\}} = \frac{\pi\lambda}{2\sqrt{2}}\left(1 + \frac{\lambda^2}{112}\right).$$

Since $\pi/(2\sqrt{2}) = 1{\cdot}11$, the pitch at the axis is about 11 per cent. larger than on the surface when λ is small.

The corresponding quantity for the vortex filaments is given by

$$s = \eta + \tfrac{1}{2}\lambda \mathrm{J}' \int_{x_1}^{x} \frac{x^2\, dx}{\sqrt{(\mathrm{J} - x^2\mathrm{J}')(\mathrm{J} - x^2\mathrm{J}' - b)}}.$$

By what has immediately gone before

$$s - \eta = \tfrac{1}{2}\lambda \mathrm{J}' \int_{x_1}^{x} \frac{30}{\lambda^4} \frac{x\, dx}{\left\{1 - \frac{\lambda^2}{28}(x^2 + 1)\right\} \sqrt{\{(1 - x^2)(x^2 - x_1^2)(x_2^2 - x^2)\}}}$$

$$= \tfrac{15}{2} \frac{\mathrm{J}'}{\lambda^3} \int_{y_1}^{y} \frac{dy}{\left(1 - \frac{\lambda^2}{28} - \frac{\lambda^2}{28}\, y\right) \sqrt{\{(1 - y)(y - y_1)(y_2 - y)\}}}$$

$$= \tfrac{15}{2} \frac{\mathrm{J}'}{\lambda^3} \int_{0}^{\theta} \frac{d\theta}{\left(1 - \frac{\lambda^2}{28} - \frac{\lambda^2}{28}\, y\right) \sqrt{1 - y}}$$

$$= \frac{15\mathrm{J}'}{2\lambda^3 \sqrt{1 - y_1}} \int_{0}^{\phi} \frac{d\phi}{\left\{1 - \frac{\lambda^2}{28}(1 + y_1) - \frac{\lambda^2}{28}(y_2 - y_1)\sin^2\phi\right\} \sqrt{(1 - k^2 \sin^2\phi)}},$$

and

$$\mathrm{J}' = \tfrac{1}{3}\lambda^2 (1 - \tfrac{1}{10}\lambda^2).$$

Therefore

$$s - \eta = \frac{5\left(1 - \frac{\lambda^2}{10}\right)}{2\lambda \sqrt{1 - y_1} \left\{1 - \frac{\lambda^2}{28}(1 + y_1)\right\}} \int_0^{\phi} \frac{d\phi}{(1 - n \sin^2\phi) \sqrt{(1 - k^2 \sin^2\phi)}},$$

where

$$n = \frac{\lambda^2}{28}(y_2 - y_1) = \frac{\lambda^2}{28} \sqrt{(1 - 4c)}.$$

Thus

$$s = \frac{\lambda}{2} \sqrt{1 - y_1}\, \mathrm{E}(k \,.\, \phi) + \frac{5 - \frac{9}{28}\lambda^2 + \frac{5\lambda^2 y_1}{28}}{2\lambda \sqrt{1 - y_1}}\, \Pi(-n, k, \phi) . \quad . \quad . \quad (38).$$

At the equatorial axis $n = 0$, $k = 0$; $\Pi = \pi/2$ for a half turn.

Thus the pitch at the equatorial axis is

$$= \frac{\pi\lambda}{2}\sqrt{1 - y_0} + \frac{5 - \frac{9}{28}\lambda^2 + \dfrac{5\lambda^2 y_0}{28}}{2\lambda\sqrt{1 - y_0}}\,\pi,$$

and

$$y_0 = \tfrac{1}{2}\left(1 - \frac{\lambda^2}{112}\right).$$

Therefore

$$\text{Pitch} = \frac{\pi}{2}\left\{\frac{\lambda}{\sqrt{2}}\left(1 + \frac{\lambda^2}{112}\right) + \frac{\sqrt{2}}{\lambda}\left(5 - \tfrac{13}{56}\lambda^2\right)\left(1 - \frac{\lambda^2}{112}\right)\right\}$$

$$= \frac{\pi}{2\lambda\sqrt{2}}\left\{\lambda^2 + 2\left(5 - \tfrac{31}{112}\lambda^2\right)\right\}$$

$$= \frac{5\pi}{\lambda\sqrt{2}}\left(1 + \tfrac{5}{112}\lambda^2\right).$$

If $\lambda = 0$, the pitch is ∞, as it clearly ought to be, since all the vortex filaments then lie along parallels.

The Form of the Spirals near an Equatorial Axis.

The meridian sections of a current sheet near an axis will evidently in general be elliptic. To find η it is therefore necessary to determine for an ellipse the value of

$$\int \frac{dr}{\cos\theta}.$$

The following general theorem enables us easily to do this. Transfer the origin to any point O′ in the equatorial plane, at a distance c; and let the new polar co-ordinates of a point P be $r'.\theta'$, corresponding to $r.\theta$. Also let $x.y$ denote the Cartesian co-ordinates referred to O′. Then

$$r^2 = r'^2 + c^2 + 2cx,$$

$$rdr = r'dr' + cdx,$$

$$\int \frac{dr}{\cos\theta} = \int \frac{rdr}{r\cos\theta} = \int \frac{r'dr' + cdx}{r'\cos\theta'} = \int \frac{dr'}{\cos\theta'} + c\int \frac{dx}{y}.$$

For the spirals near the axis the point of interest is to determine the angular pitch. Now clearly for a complete ellipse, whose axes are parallel and perpendicular to the equatorial plane, and whose centre is at O′

$$\int \frac{dr'}{\cos\theta'} = 0.$$

Further, if the axes are α, β, respectively in the equatorial plane and perpendicular to it,

$$x = \alpha \sin \theta, \qquad y = \beta \cos \theta,$$

where θ is the excentric angle of a point on it. Hence,

$$\int \frac{dx}{y} = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\alpha \cos \theta \, d\theta}{\beta \cos \theta} = \pi \frac{\alpha}{\beta} .$$

Therefore,

$$\eta \text{ (for half-turn)} = \frac{\pi \alpha c}{\beta} \cdot \frac{\lambda}{2a} ,$$

or,

$$\text{angular pitch} = \frac{\pi \alpha c}{\beta} \frac{\lambda}{a} .$$

To apply this, it is necessary to determine the form of the current sheets near the axis.

Let the co-ordinates of the equatorial axis be c, o.

The equation to a current sheet is

$$\left\{ \mathrm{J} \left(\frac{\lambda r}{a} \right) - \frac{r^2}{a^2} \mathrm{J}(\lambda) \right\} \sin^2 \theta = \text{constant},$$

or,

$$\frac{\rho^2}{r^2} \left\{ \mathrm{J} \left(\frac{\lambda r}{a} \right) - k \left(\frac{\lambda r}{a^2} \right)^2 \right\} = \text{constant}.$$

$k = \mathrm{J}(\lambda)/\lambda^2$, and r, ρ are nearly $= c$.

Denote $\mathrm{J} \frac{\lambda r}{a} - k \left(\frac{\lambda r}{a} \right)^2$ by f, and suppose it expressed in terms of x, y co-ordinates. Refer to O′.

Then $x = c + \xi$, $y = o + \eta$, where ξ, η are small. Hence, if f now denote the value at O′,

$$\frac{(c + \xi)^2}{(c + \xi)^2 + \eta^2} \left[f + \frac{df}{dx} \cdot \xi + \frac{df}{dy} \eta + \tfrac{1}{2} \left\{ \xi^2 \frac{d^2 f}{dx^2} + 2 \xi \eta \frac{d^2 f}{dx \, dy} + \eta^2 \frac{d^2 f}{d\eta^2} \right\} \right] = \text{constant},$$

$$\frac{df}{dx} = \frac{df}{dr} \frac{x}{r} , \qquad \frac{df}{dy} = \frac{df}{dr} \frac{y}{r} ,$$

and $df/dr = o$, for c is given by this equation.

Denote df/dr by f', $d^2 f/dr^2$ by f''. Then

$$\frac{d^2 f}{dx^2} = \left(\frac{1}{r} - \frac{x^2}{r^3} \right) f' + \frac{x^2}{r^2} f'' = \frac{x^2}{r^2} f'' = f'', \text{ since } x = r \text{ to 1st order},$$

$$\frac{d^2 f}{dx \, dy} = - \frac{xy}{r^3} f' + \frac{xy}{r^2} f'' = 0, \text{ since } y = 0 \text{ to 1st order},$$

$$\frac{d^2 f}{dy^2} = \left(\frac{1}{r} - \frac{y^2}{r^2} \right) f' + \frac{y^2}{r^2} f'' = 0.$$

Hence

$$\left(1 - \frac{\eta^2}{c^2}\right)(f + \tfrac{1}{2}\xi^2 f'') = \text{constant}.$$

The constant is nearly $f = f - \alpha$ say; then

$$-\frac{f}{c^2}\eta^2 + \tfrac{1}{2}f''\xi^2 = -\alpha,$$

or the curve is the ellipse

$$\frac{\xi^2}{2\alpha / -f''} + \frac{\eta^2}{\alpha c^2/f} = 1.$$

Hence the angular pitch $= \dfrac{\pi\lambda c}{a}\sqrt{\dfrac{2f}{-c^2 f''}} = \dfrac{\pi\lambda}{a}\sqrt{-\dfrac{2f}{f''}}$.

Now

$$f'' = \frac{d^2}{dr^2}\left\{\frac{\mathrm{J}\,\lambda r}{a} - \frac{r^2}{a^2}\mathrm{J}\lambda\right\} = \frac{\lambda^2}{a^2}\frac{d^2}{dy^2}\left\{\mathrm{J}y - y^2\frac{\mathrm{J}(\lambda)}{\lambda^2}\right\}$$

$$= \frac{\lambda^2}{a^2}\left\{\frac{d^2\mathrm{J}}{dy^2} - 2\frac{\mathrm{J}\lambda}{\lambda^2}\right\} = \frac{\lambda^2}{a^2}\left\{\left(\frac{2}{y^2} - 1\right)\mathrm{J} - \frac{2\mathrm{J}}{\lambda^2}\right\} = \frac{\lambda^2}{a^2}\left(\frac{2f}{y^2} - \mathrm{J}y\right).$$

The angular pitch of the stream-lines is therefore

$$\pi\sqrt{\left\{\frac{2f}{\mathrm{J}(y) - \dfrac{2f}{y^2}}\right\}} = \frac{\pi}{p},$$

where

$$p^2 = \tfrac{1}{2}\frac{\mathrm{J}(y)}{\mathrm{J}(y) - y^2\dfrac{\mathrm{J}(\lambda)}{\lambda^2}} - \frac{1}{y^2}.$$

Now y is determined by $\mathrm{J}(y) - y\sin y + 2y^2\dfrac{\mathrm{J}(\lambda)}{\lambda^2} = 0$, therefore

$$p^2 = \tfrac{1}{2}\frac{\sin y - 2y\dfrac{\mathrm{J}(\lambda)}{\lambda^2}}{\sin y - 3y\dfrac{\mathrm{J}(\lambda)}{\lambda^2}} - \frac{1}{y^2}$$

$$= \tfrac{1}{2} - \frac{1}{y^2} + \tfrac{1}{2}\frac{1}{\dfrac{\sin y}{y\dfrac{\mathrm{J}\lambda}{\lambda^2}} - 3}$$

(Note for λ_1 aggregates $p^2 = \tfrac{1}{2} - \dfrac{1}{y^2}$).

When the value of λ is fairly large we substitute for y from equation (36)

$$y = m\pi + z,$$

where

$$z = \frac{b-1}{\alpha} + \frac{(b-1)(b+2)(b+5)}{6\alpha^3},$$

$$\alpha \equiv m\pi, \qquad b \equiv 2(-)^m\alpha^2\frac{\mathrm{J}(\lambda)}{\lambda^2}.$$

Therefore

$$p^2 = \tfrac{1}{2} - \frac{1}{(\alpha+z)^2} + \tfrac{1}{2}\frac{1}{(-1)^m\dfrac{\sin z}{(\alpha+z)\dfrac{b}{2(-)^m\alpha^2}} - 3}$$

$$= \tfrac{1}{2} - \frac{1}{\alpha^2} + \tfrac{1}{2}\frac{1}{\dfrac{2\alpha^2}{\alpha+z}\cdot\dfrac{z-\frac{1}{6}z^3}{b} - 3}$$

$$= \tfrac{1}{2} - \frac{1}{\alpha^2} + \tfrac{1}{2}\frac{1}{\dfrac{2\alpha z}{b}\left(1-\dfrac{z}{\alpha}+\dfrac{z^2}{\alpha^2}\right)\left(1-\dfrac{z^2}{6}\right) - 3}$$

$$= \tfrac{1}{2} - \frac{1}{\alpha^2} + \tfrac{1}{2}\frac{1}{\dfrac{2(b-1)}{b}\left(1+\dfrac{\overline{b+2}\,\overline{b+5}}{6\alpha^2}\right)\left(1-\dfrac{(b-1)^2}{6\alpha^2}\right)\left(1-\dfrac{b-1}{\alpha^2}\right) - 3}$$

$$= \tfrac{1}{2} - \frac{1}{\alpha^2} + \tfrac{1}{2}\frac{1}{\dfrac{2(b-1)}{b}\left\{1+\dfrac{1}{6\alpha^2}(\overline{b+2}\,\overline{b+5} - \overline{b-1}|^2 - 6\,\overline{b-1})\right\} - 3}$$

$$= \tfrac{1}{2} - \frac{1}{\alpha^2} + \tfrac{1}{2}\frac{1}{\dfrac{b-1}{b}\left(2+\dfrac{b+5}{\alpha^2}\right) - 3}$$

$$= \tfrac{1}{2} - \frac{1}{\alpha^2} + \tfrac{1}{2}\frac{b}{-(b+2)+\dfrac{(b-1)(b+5)}{\alpha^2}}.$$

$$p^2 = \frac{1-\dfrac{(b-1)(b+5)}{2\alpha^2}}{b+2-\dfrac{(b-1)(b+5)}{\alpha^2}} - \frac{1}{\alpha^2} = \frac{1-\dfrac{b^2+6b-1}{2\alpha^2}}{b+2-\dfrac{(b-1)(b+5)}{\alpha^2}},$$

and the pitch is

$$\pi\sqrt{\left\{\frac{b+2-\dfrac{(b-1)(b+5)}{\alpha^2}}{1-\dfrac{b^2+6b-1}{2\alpha^2}}\right\}} = \pi\sqrt{\left\{b+2+\frac{(b+2)^3-9b}{2\alpha^2}\right\}}.$$

Now

$$b = 2(-)^m\left(\frac{\alpha}{\lambda}\right)^2\mathrm{J}\lambda,$$

where

$$\lambda = \lambda_2^{(n)} - \mathrm{X}$$

and

$$\lambda_2^{(n)} = n\pi - \left(\frac{3}{n\pi}\right) - \tfrac{1}{3}\left(\frac{3}{n\pi}\right)^3 - \ldots$$

and

$$X \overset{=}{<} \pi = q\pi \text{ say,}$$

denote $n\pi$ by β; then

$$\frac{\alpha^2 J\lambda}{\lambda^2} = (-)^n \left(\frac{m}{n}\right)^2 \left(1 - \frac{3}{\beta^2} - \frac{X}{\beta}\right)^{-2} \left\{ - \frac{\sin\left(X + \frac{3}{\beta} + \frac{9}{\beta^3}\right)}{\beta\left(1 - \frac{3}{\beta^2} - \frac{X}{\beta}\right)} - \cos\left(X + \frac{3}{\beta} + \frac{9}{\beta^3}\right) \right\}$$

$$= -(-)^n \left(\frac{m}{n}\right)^2 \left(1 + \frac{2X}{\beta} + \frac{6 + 3X^2}{\beta^2}\right) \left\{ \frac{1}{\beta}\left(1 + \frac{X}{\beta}\right)\left(\sin X + \frac{3}{\beta}\cos X\right) \right.$$

$$\left. + \cos X\left(1 - \frac{9}{2\beta^2}\right) - \sin X\left(\frac{3}{\beta}\right)\right\}$$

$$= -(-)^n \left(\frac{m}{n}\right)^2 \left(1 + \frac{2X}{\beta} + \frac{6 + 3X^2}{\beta^2}\right) \left\{\cos X\left(1 - \frac{3}{2\beta^2}\right) + \sin X\left(-\frac{2}{\beta} + \frac{X}{\beta^2}\right)\right\},$$

therefore

$$b + 2 = 2\left[1 - (-)^{m+n}\left(\frac{m}{n}\right)^2\left\{\left(1 + \frac{2X}{\beta} + \frac{9 + 6X^2}{2\beta^2}\right)\cos X - \left(\frac{2}{\beta} + \frac{3X}{\beta^2}\right)\sin X\right\}\right].$$

For very large values of λ we may neglect powers of $\frac{1}{\beta}$, and then

$$\text{pitch} = \pi\sqrt{2}\left\{1 - (-)^{m+n}\left(\frac{m}{n}\right)^2 \cos X\right\}^{\frac{1}{2}}.$$

For the outside shell $m = n$,

$$\text{pitch} = 2\pi \sin \tfrac{1}{2}X.$$

Thus in the case of the λ_2 aggregates the pitch of the outer layer is very small.

If we number the shells backward from the outside, we write $n - p + 1$ for m, and the pitch is

$$\pi\sqrt{2}\left\{1 + (-)^p\left(\frac{n - p + 1}{n}\right)^2 \cos X\right\}^{\frac{1}{2}}.$$

It is seen, therefore, that there are two series of shells in aggregates of large λ, one in which the pitches increase as we pass inwards, and an alternate series in which it decreases. If λ lies between a λ_1 and a λ_2 parameter ($\lambda_2 > \lambda_1$), the outer series belongs to the first category. If λ lies between a λ_2 and a λ_1 value ($\lambda_1 > \lambda_2$), the opposite is the case. In other words, if the parametral point P in fig. 2, Plate 1, lie above the line of abscissæ, the outside layer has a very small pitch, and those of alternate shells increase as we go to the centre. If P lie below the opposite is the case.

The vortex spirals are given by

$$\varsigma - \eta = \frac{\lambda}{2} J(\lambda) \int_{x_1}^{x} \frac{x^2\, dx}{\{(J - x^2 J')(J - x^2 J' - b)\}^{\frac{1}{2}}}.$$

Write $J - x^2 J' = f$.

Let x_0 be the value of x at the axis, so that $f' = 0$ when $x = x_0$.

For points near the axis,

$$x = x_0 + \xi,$$

$$f(x) = f + af'.\xi + \tfrac{1}{2}a^2 f''.\xi^2 = f + \tfrac{1}{2}a^2 f''.\xi^2,$$

and

$$\varsigma - \eta = \tfrac{1}{2}\frac{\lambda J'}{a^2}\int_{-\xi_1}^{\xi_1}\frac{(x_0 + \xi)^2\,d\xi}{(\tfrac{1}{2}f'')\left\{\left(\frac{2f}{a^2 f''} + \xi^2\right)\left(2\frac{f - b}{a^2 f''} + \xi^2\right)\right\}^{\frac{1}{2}}},$$

where $(\tfrac{1}{2}f'')$ means the positive value of $\tfrac{1}{2}f''$.

To the first order,

$$\xi_1^2 = \xi_2^2 = -2\frac{f - b}{a^2 f''},$$

$$\varsigma - \eta = \frac{\lambda J'}{a^2 (f'')}\int_{-\xi_1}^{\xi_1}\frac{(x_0 + \xi)^2\,d\xi}{\left\{(\xi_1^2 - \xi^2)\left(-\frac{2f}{a^2 f''} - \xi^2\right)\right\}^{\frac{1}{2}}}.$$

Hence for the total pitch,

$$\varsigma - \eta = \frac{2\lambda J'}{a^2 (f'')}\int_{-\xi_1}^{+\xi_1}\frac{(x_0^2 + \xi^2)\,d\xi}{\left\{(\xi_1^2 - \xi^2)\left(-\frac{2f}{a^2 f''} - \xi^2\right)\right\}^{\frac{1}{2}}}.$$

Put

$$\xi = \xi_1 \sin\theta.$$

$$\varsigma - \eta = \frac{4\lambda J'}{a^2 (f'')}\int_0^{\frac{1}{2}\pi}\frac{(x_0^2 + \xi_1^2 \sin^2\theta)\,d\theta}{\sqrt{\left(-\frac{2f}{a^2 f''} - \xi_1^2 \sin^2\theta\right)}}$$

or writing

$$k^2 = -\frac{a^2 f'' \xi_1^2}{2f}$$

$$\varsigma - \eta = \frac{4\lambda J'}{a\sqrt{(-2ff'')}}\int_0^{\frac{1}{2}\pi}\frac{x_0^2 - \frac{2f}{a^2 f''} + \frac{2f}{a^2 f''} + \xi_1^2 \sin^2\theta}{\sqrt{(1 - k^2 \sin^2\theta)}}\,d\theta$$

$$= \frac{4\lambda J'}{a\sqrt{(-2ff'')}}\left\{\left(x_0^2 - \frac{2f}{a^2 f''}\right)\mathrm{F} + \frac{2f}{a^2 f''}\mathrm{E}\right\}.$$

At the axis itself $k^2 = 0$.

$$\varsigma - \eta = \frac{4\lambda J' x_0^2}{a\sqrt{(-2ff'')}}\cdot\frac{\pi}{2}$$

$$\varsigma = \pi\lambda\sqrt{-\frac{2f}{a^2 f''}} + 2\pi\lambda\frac{J' x_0^2}{a\sqrt{(-2ff'')}}$$

$$= \frac{2\pi\lambda}{a\sqrt{(-2ff'')}}\{(f) + x_0^2 J'\}$$

$$= \frac{2\pi\lambda J(y)}{a\sqrt{(-2ff'')}} \quad \text{if} \quad J(y) > y^2\frac{J\lambda}{\lambda^2}$$

$$= \frac{2\pi J y}{\sqrt{2f\left(Jy - \frac{2f}{y^2}\right)}} \quad \text{if} \quad J(y) > y^2\frac{J(\lambda)}{\lambda}$$

or

$$\varsigma - \eta = \frac{2\pi}{\sqrt{2f\left(\mathrm{J}y - \frac{2f}{y^2}\right)}}\left(\frac{2y^2\mathrm{J}\lambda}{\lambda^2} - \mathrm{J}y\right), \quad \text{if} \quad \mathrm{J}y < \frac{y^2\mathrm{J}\lambda}{\lambda^2},$$

where $y \equiv \lambda x_0/a$.

Spirals on the bounding surface, or interface between two shells. This is the case where the transformation (Eq. 26) fails. Taking first the stream spirals

$$d\eta = \frac{\lambda}{a} \cdot \frac{\psi}{\rho} \frac{ds}{\frac{d\psi}{dn}}.$$

On a spherical boundary this is zero, except for the λ_2 aggregates, in which, however, there is no flow at all. The other part of the stream surface is the portion up the polar axis. Here $ds = dr$ and $dn = rd\theta$. Therefore

$$\text{Twist on axis alone} = \frac{2\lambda}{a}\int_0^a \frac{\psi dr}{\rho \frac{d\psi}{rd\theta}} (\theta = 0)$$

$$= \frac{\lambda}{a}\int_0^a dr = \lambda.$$

There is no twist on the spherical boundary. Hence

Angular pitch of stream spiral $= \lambda$.

Next for the vortex spirals. Here there are two portions as in the former case—the polar axis, and the spherical boundary.

$$d\varsigma = \frac{\lambda}{a} \cdot \frac{\mathrm{J}\sin^2\theta}{\rho \frac{d}{dn}\{(\mathrm{J} - x^2\mathrm{J}')\sin^2\theta\}} ds.$$

Hence, supposing at present we are dealing with a singlet only

$$\varsigma = \frac{2\lambda}{a}\int_0^a \frac{1}{2}\frac{\mathrm{J}}{\mathrm{J} - x^2\mathrm{J}'} \cdot dr + \frac{2\lambda}{a}\int_0^{\frac{1}{2}\pi} \frac{\mathrm{J}\lambda \,.\, a d\theta}{a \sin\theta \frac{d}{dr}(\mathrm{J} - x^2\mathrm{J}')_{r=a}}$$

$$= \lambda\int_0^1 \frac{\mathrm{J}}{\mathrm{J} - x^2\mathrm{J}'} dx + \frac{2\lambda\mathrm{J}\lambda}{\frac{d}{dx}(\mathrm{J} - x^2\mathrm{J}')_{x=1}}\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sin\theta}.$$

Both these integrals become infinite at the poles. We must therefore treat this part separately.

$$\varsigma = \lambda\int_0^{1-\xi} \frac{\mathrm{J}}{\mathrm{J} - x^2\mathrm{J}'} dx + \frac{2\lambda\mathrm{J}(\lambda)}{\lambda\sin\lambda - 3\mathrm{J}(\lambda)}\int_\alpha^{\frac{1}{2}\pi} \frac{d\theta}{\sin\theta} + 2\frac{\lambda}{a}\int_{\xi_1\alpha'}^{\xi_1'\alpha} \frac{\mathrm{J}\sin^2\theta\, ds}{\rho\left\{\left(\frac{d\psi'}{dr}\right)^2 + \left(\frac{d\psi'}{rd\theta}\right)^2\right\}^{\frac{1}{2}}},$$

in which ξ, α are small, and in the third integral, x and θ are nearly 1, 0 respectively. Let $x = 1 - \xi$, $\sin\theta = \eta$. Then near the pole the rectangular co-ordinates of a point referred to the pole are connected by (if $f \equiv \mathrm{J} - x^2\mathrm{J}'$)

$$f \sin^2\theta = \text{small constant} = \beta \text{ (say)},$$

$$-\frac{df}{dx}\cdot\xi\sin^2\theta = \beta, \qquad \xi\eta^2 = -\frac{\beta}{\dfrac{df}{dx}} = \gamma^3 \text{ say},$$

so that when $\xi = \eta$ each $= \gamma$.

The third integral is

$$2\lambda\int\frac{\mathrm{J}\sin^2\theta\sqrt{(d\xi^2 + d\eta^2)}}{r\sin\theta\,\{(df/dr)^2\sin^4\theta + 4\sin^2\theta\cos^2\theta\,(f/r)^2\}^{\frac{1}{2}}}$$

$$= 2\lambda\int\frac{\mathrm{J}\sqrt{(d\xi^2 + d\eta^2)}}{(1-\xi)\left\{(df/dx)^2\sin^2\theta + 4\cos^2\theta\left(\dfrac{df}{xdx}\cdot\xi\right)^2\right\}^{\frac{1}{2}}}$$

$$= \frac{2\lambda\mathrm{J}'}{df/dx}\left[\int_{\xi_1}^{\gamma}\sqrt{\left(\frac{1 + (d\eta/d\xi)^2}{\eta^2 + 4\xi^2}\right)}d\xi + \int_{\gamma}^{\eta_1}\sqrt{\left(\frac{1 + (d\xi/d\eta)^2}{\eta^2 + 4\xi^2}\right)}d\eta\right].$$

The curve is given by

$$\xi\eta^2 = \gamma^3.$$

Therefore

$$\frac{d\eta}{d\xi} = -\frac{\eta^2}{2\xi\eta} = -\tfrac{1}{2}\frac{\eta}{\xi} = -\tfrac{1}{2}\left(\frac{\gamma}{\xi}\right)^{3/2} = -\tfrac{1}{2}\left(\frac{\eta}{\gamma}\right)^3.$$

Therefore

$$\text{Integral} = \frac{2\lambda\mathrm{J}'}{df/dx}\left[\int_{\xi_1}^{\gamma}\sqrt{\left(\frac{1 + \frac{1}{4}(\gamma/\xi)^3}{\gamma^3/\xi + 4\xi^2}\right)}d\xi + \int_{\gamma}^{\eta_1}\sqrt{\left(\frac{1 + 4\gamma^6/\eta^6}{\eta^2 + 4\gamma^6/\eta^4}\right)}d\eta\right]$$

$$= \frac{2\lambda\mathrm{J}'}{df/dx}\left[\tfrac{1}{2}\int_{\xi_1}^{\gamma}\sqrt{\left(\frac{4\xi^3 + \gamma^3}{4\xi^3 + \gamma^3}\right)}\frac{d\xi}{\xi} + \int_{\gamma}^{\eta_1}\sqrt{\left(\frac{\eta^6 + 4\gamma^6}{\eta^6 + 4\gamma^6}\right)}\cdot\frac{d\eta}{\eta}\right]$$

$$= \frac{2\lambda\mathrm{J}'}{df/dx}\left[\tfrac{1}{2}\log\frac{\gamma}{\xi_1} + \log\frac{\eta_1}{\gamma}\right] = \frac{\lambda\mathrm{J}'}{df/dx}\log\frac{\eta_1^2}{\gamma\xi_1}.$$

The second integral is

$$\frac{2\lambda\mathrm{J}'}{df/dx}\int_{\theta}^{\frac{1}{2}\pi}\frac{d\theta}{\sin\theta} = \frac{2\lambda\mathrm{J}'}{df/dx}\left[\log\tan\frac{\theta}{2}\right]_{\theta}^{\frac{1}{2}\pi}$$

$$= \frac{\lambda\mathrm{J}'}{df/dx}\log\frac{1+\cos\theta}{1-\cos\theta} = \frac{\lambda\mathrm{J}'}{df/dx}\log\frac{(1+\cos\theta)^2}{\sin^2\theta},$$

and θ is nearly $= 0$. Therefore

$$\text{Second Integral} = \frac{\lambda\mathrm{J}'}{df/dx}\log\frac{4}{\eta_1^2},$$

and the second and third together

$$= \frac{\lambda\mathrm{J}'}{df/dx}\log\frac{4}{\gamma\xi_1}.$$

The first integral is

$$\lambda \int_0^{1-\xi_1} \frac{\mathrm{J}}{\mathrm{J} - x^2\mathrm{J}'} dx = \lambda + \lambda\mathrm{J}' \int_0^{1-\xi_1} \frac{x^2\, dx}{\mathrm{J} - x^2\mathrm{J}'} .$$

Now $\mathrm{J} - x^2\mathrm{J}' = x^2 (1 - x^2) \mathrm{F}(x)$, where $\mathrm{F}(x)$ is finite for x between 0 and 1 and does not vanish. Hence

$$\begin{aligned} \text{Int.} &= \lambda + \lambda\mathrm{J}' \int_0^{1-\xi_1} \frac{dx}{(1 - x)(1 + x)\mathrm{F}x} \\ &= \lambda + \lambda\mathrm{J}' \int_0^{1-\xi_1} \left\{ - \frac{1}{df/dx} \frac{dx}{1 - x} + \ldots \right\} \\ &= \lambda + \frac{\lambda\mathrm{J}'}{df/dx} \log \xi_1 + \text{finite quantity}, \end{aligned}$$

Therefore

$$\mathrm{s} = \lambda + \frac{\lambda\mathrm{J}'}{df/dx} \log \frac{4}{\gamma} + \text{finite}.$$

$$\mathrm{s} = \lambda + \frac{\lambda\mathrm{J}'}{\lambda \sin \lambda - 3\mathrm{J}'} \log \frac{4a\sqrt{2}}{s} + \lambda\mathrm{J}' \int_0^1 \left\{ \frac{x^2}{\mathrm{J} - x^2\mathrm{J}'} + \frac{1}{\frac{df}{dx}(1 - x)} \right\} dx,$$

where s is the distance from the pole of the point at which the stream sheet ψ cuts a line joining the pole to a point on the equator. The angular pitch is therefore infinite at the surface owing to the filaments being parallel to the equator at points close to the pole.

25. *Graphical Methods.*—The graphical construction indicated in § 17 affords a very convenient method of obtaining a general qualitative view of the properties of these aggregates. It serves also for a rough quantitative one, and at least gives for many determinations the rough starting point which is always the most troublesome obstacle in numerical approximations. It may be well, therefore, here, to collect and enlarge on what has gone before in this respect.

The first thing is to trace on a large scale the curve $y = \mathrm{J}(\lambda)$ where λ is the abscissa. This is very easily done, since J is expressed in simple functions which are tabulated. The curve is drawn for the first three undulations in fig. (2), Plate (1). Now λ determines completely the *nature* of the aggregate (except its volume and its intensity). The point P on the J curve, corresponding to λ, we will call the parametral point. Draw through P a parabola touching the axis at the origin. For all points beyond the first few undulations a circle will suffice, or the curve drawn by a thin lath bent to touch the axis at O and to pass through P. If x denote r/a, λx will correspond to a point on the J curve between O and P. If P_1, P_2 denote the corresponding points on the J curve and the parabola, the value of ψ in the aggregate at the point $(r = xa, \theta)$ is given by $\mathrm{P}_1\mathrm{P}_2 \sin^2\theta$ (note $\mathrm{P}_2\mathrm{P}_1$ will be negative). The velocity of propagation will depend on the angle at which the parabola and curve intersect at P (see fig. 3, § 20). If they touch, the angle is zero, and the translation velocity zero. In fact the parameters of the points are the λ_2 values. We will call them the Q points. They are easily formed by fixing a lath at O and bending it to touch successive loops of the J curve. It is easy to do this correct to two decimal places, when numerical calculation will carry it to any degree of approximation desired.

The points where the J curve cuts the axis of x correspond to the λ_1 parameters. We will call them the R points.

Denote the points where the parabola through P cuts the J curve again by the letters p. These points give the sizes of the shells into which the aggregate divides. If ON be the abscissa of any such point, $\lambda x = \text{ON}$, and $r = \frac{\text{ON}}{\lambda} . a$ gives the radius of the corresponding interface between two shells. It is evident at once from the construction that the thicknesses of the shells, as we pass in or out, are alternately greater and less—that there are two categories, in one of which the thickness increases as we pass in, and an alternate series in which it decreases. There will be, however, some irregularity in the two inner components.

The position of the equatorial axes is determined by those abscissæ, for which the tangents to the J curve and the parabola are parallel. They are easily recognized by the eye, and thus a starting point for calculation is readily obtained. The difference of ordinates of these points (P_1P_2) is proportional to the secondary circulations of the corresponding shells. In fact, when multiplied by $\pi\mu/(\text{S}i\lambda - \sin\lambda)$, the products give the values of those constants. It is therefore clear from the figure that these circulations are in opposite directions alternately, and that we get two alternate series of ascending and descending values.

The function $\text{S}(\lambda) \equiv \text{S}i\lambda - \sin\lambda$ denotes the area between the J curve and the axis of x up to the point λ. It is clear, therefore, that it has its maximum values at the odd λ_1 points, and its minimum at the even ones.

The tracing of the current sheets is particularly easy from the fact that they are given by functions of the form

$$\psi = \text{F}(r) . \sin^2\theta.$$

Let

$$f(r) \equiv \text{J}\left(\frac{\lambda r}{a}\right) - \frac{r^2}{a^2}\text{J}(\lambda),$$

and let ψ_0 and r_0 denote values at the equatorial axis (*i.e.*, ψ_0 a numerical maximum). Then

$$\frac{\psi}{\psi_0} = \frac{f(r)}{f(r_0)}\sin^2\theta, \qquad \sin\theta = \sqrt{\frac{\psi}{\psi_0} . \frac{f(r_0)}{f(r)}}.$$

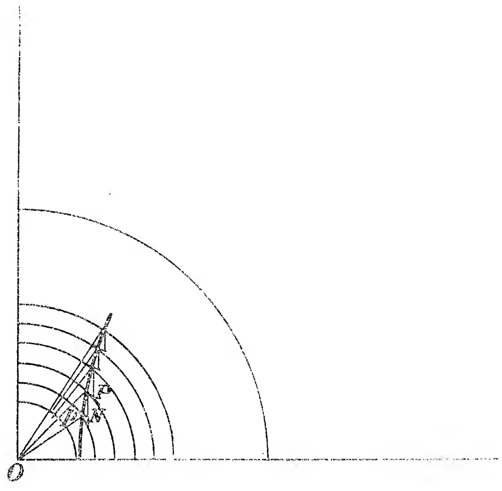
On squared paper, draw a series of circles, radii sub-multiples of a, say at intervals of $\cdot 05a$ or $\cdot 1a$, also the circle $r = r_0$. This last circle has the property that all the current sheets cut it at right angles.

Let us trace first one sheet (say $\psi = \cdot 1\psi_0$). We do this by tabulating the values of $\sin\theta$ for values of r, corresponding to the series of circles drawn. Now mark on the bounding circle $(r = a)$ points whose abscissæ are those tabulated values (which is done at once on the squared paper). Mark the points where the radii vectores to these points cut the corresponding circles. Join these points by a continuous curve, and the shape of the particular ψ curve is obtained ; call it the ψ_1 sheet. This first curve should be obtained with care and as much accuracy as possible. We may now proceed to draw from this as many of the other sheets as we please. Suppose we want to draw the curve $\psi = k \, . \, \psi_0$. We set a pair of proportional compasses (or any similar method) to the ratio $\sqrt{10k}$. Suppose the ψ_1 cuts any particular circle at P, set the short legs of the compasses to its abscissa. Turn it round and find the point on the *same* circle whose abscissa is the new value. Proceed thus with the other circles and the sheet is rapidly traced. Although this may appear cumbrous in stating, it is very expeditious in practice, and with a moderate amount of care very accurate.

Having traced the ψ curves, we may now easily trace the projections of the stream lines, for these are given by

$$\eta = \int \frac{dr}{\cos\theta} = \Sigma \text{PN} \quad \text{(see fig. 4).}$$

Fig. 4.



26. It will be interesting to go into further details for a few cases, and for this purpose we take the first two aggregates of the λ_2 and λ_1 families.

The distinguishing feature of the λ_2 types, is that the aggregates are at rest in the surrounding fluid. The distinguishing feature of the λ_1 types is that the vortex and stream lines are coincident.

λ_2 aggregates. Here

$$\text{U} = 0, \quad \text{M} = \frac{m\mu}{15} \frac{-\lambda \sin\lambda}{Si\lambda - \sin\lambda},$$

$$\text{E} = \tfrac{2}{45} \pi\mu^2 a \frac{\lambda^2 \sin^2\lambda}{(Si\lambda - \sin\lambda)^2} = \tfrac{7}{9} \cdot \frac{\lambda^2 \sin^2\lambda}{(Si\lambda - \sin\lambda)^2} \text{E}_0,$$

where E_0 is the energy of a Hill's aggregate of equal volume and circulation.

The first parameter is $\lambda_2^{(1)} = 5{\cdot}7637 = 330^\circ\ 14'$.

The equatorial axis has a radius $= {\cdot}5130a$,

$$\begin{aligned} \mathrm{M} &= {\cdot}0985 m\mu \\ \mathrm{E} &= 1{\cdot}6979 \mathrm{E}_0 \\ \nu &= 2{\cdot}1020\mu . \end{aligned}$$

Angular pitch of stream lines at surface = 330° 14′.
,, ,, ,, at axis = 334° 58′.
,, ,, vortex lines at axis = 267°.

The forms of the current sheets (ψ) are drawn in fig. 2, Plate 2. The projections of the stream lines in fig. 3, Plate 2. These latter were determined by the graphical method described above.

The second λ_2 parameter is

$$\lambda_2^{(2)} = 9{\cdot}0950 = 3\pi - 18^\circ\ 53'\ 40''.$$

The equatorial axes are given by

$$\begin{aligned} \lambda_2 x &= 2{\cdot}6616 \text{ and } 6{\cdot}2718, \\ \text{or } r &= {\cdot}2926a \text{ and } {\cdot}6896a. \end{aligned}$$

Radius of internal nucleus $= {\cdot}4694a$,

$$\begin{array}{rlrl} \mathrm{M} = & -\ {\cdot}1459 m\mu & \mathrm{M}_1 = & {\cdot}08904 m_1\mu_1 \\ \mathrm{E} = & 3{\cdot}727\mathrm{E}_0 & \mathrm{M}_2 = & {\cdot}1890 m_2\mu_2 \\ \mu_1 = & 1{\cdot}9403\mu & \nu_1 = & 1{\cdot}1747\mu_1 \\ \mu_2 = & -\ {\cdot}9403\mu & \nu_2 = & 3{\cdot}6415\mu_2 . \\ \mu_1/\mu_2 = & -\ 2{\cdot}063 & & \end{array}$$

Total angular pitch of stream lines outside = 521° 6′.
,, ,, ,, inside nucleus = 244° 37′.
,, ,, ,, outer shell = 276° 29′.
Angular pitch of stream lines at 1st axis = 284° 21′.
,, ,, ,, 2nd axis = 320° 9′.
,, ,, vortex lines at 1st axis = 308° 48′.
,, ,, ,, 2nd axis = 422° 11′.

The λ_1 aggregates. Here

$$\mathrm{U} = \frac{\mu}{3a} \frac{-\sin\lambda}{\mathrm{S}i\lambda - \sin\lambda} = \tfrac{5}{3} \frac{-\sin\lambda}{\mathrm{S}i\lambda - \sin\lambda} \mathrm{U}_0,$$

$$\mathrm{M} = m\mu \frac{-\sin\lambda}{\lambda\,(\mathrm{S}i\lambda - \sin\lambda)},$$

$$\mathrm{E} = \tfrac{4}{9}\pi\mu^2 a \frac{\sin^2\lambda}{(\mathrm{S}i\lambda - \sin\lambda)^2} = \tfrac{70}{9} \frac{\sin^2\lambda}{(\mathrm{S}i\lambda - \sin\lambda)^2} \mathrm{E}_0,$$

where U_0 is the velocity of translation of a HILL's aggregate of the same volume and circulation.

The stream pitch of these aggregates at the axis takes a very simple form, viz.,

$$\eta = \frac{\pi}{\sqrt{\left(\frac{1}{2} - \frac{1}{y^2}\right)}}.$$

The first λ_1 root is $\lambda_1^{(1)} = 4{\cdot}4935 = 257° \, 27' \, 30.''$

The equatorial axis has a radius $= {\cdot}6106a$.

$$\begin{aligned} U &= {\cdot}6189U_0, & M &= {\cdot}0826m\mu, \\ E &= 10{\cdot}724E_0, & \nu &= {\cdot}8016\mu. \end{aligned}$$

Angular pitch of stream lines at surface = 257° 27′ 30″.
,, ,, ,, axis = 297° 4′.

The forms for the stream sheets ψ are shown in fig. 4, Plate 2. It is to be noticed that there is a considerable difference between the angular pitches outside and on the axis, whereas in the $\lambda_2^{(1)}$ aggregate they were very nearly the same.

The second λ_1 parameter is $\lambda_1^{(2)} = 7{\cdot}7253 = 450° - 7° \, 22' \, 27''$.

The equatorial axes are given by

$$y = \lambda_1 x = 2{\cdot}7437 \quad \text{and} \quad 6{\cdot}1168,$$

or,

$$r = {\cdot}3552a \quad \text{and} \quad {\cdot}7918a.$$

Radius of internal nucleus $= {\cdot}5816a$.

$$\begin{aligned} U &= -\,3{\cdot}094U_0 & \mu_1/\mu_2 &= -\,1{\cdot}2500 \\ M &= -\,{\cdot}2403m\mu & M_1 &= {\cdot}0826m_1\mu_1 \\ E &= 26{\cdot}803E_0 & M_2 &= {\cdot}1005m_2\mu_2 \\ \mu_1 &= 4{\cdot}9740\mu & \nu_1 &= 1{\cdot}2707\mu_1 \\ \mu_2 &= -\,3{\cdot}9740\mu & \nu_2 &= 1{\cdot}5135\mu_2 \end{aligned}$$

Total angular pitch of outside = 442° 37′ 33″.
,, ,, on inside nucleus = 257° 27′ 30″.
,, ,, ,, outer shell = 185° 10′.
Angular pitch at inner axis = 297° 4′.
,, ,, outer ,, = 261° 39′.

In all the λ_1 aggregates the expression for the angular pitch at an axis is

$$\frac{\pi}{\sqrt{\left(\frac{1}{2} - \frac{1}{y^2}\right)}}.$$

Hence, when λ_1 is large, the outer layers have their pitches at the axes about $\pi\sqrt{2} = 254^\circ\ 31'$.

Fig. 5, Plate 2, shows the relative positions of the shells and axes for the λ_2^2 and λ_1^2 aggregates. The thin lines belong to the λ_2, the dotted to the λ_1. A, A are the position of the λ_2 equatorial axes. B, B those of the λ_1.

27. In the preceding investigation we find doublets, triplets, &c., naturally arising. We may have also built-up systems consisting of monads, dyads, &c., as in the cases developed in the previous section. Each element of a poly-ad may consist again of singlets, doublets, &c. I do not propose now to develop this theory of multiple combination to any length, but merely to draw attention to it, and to determine the necessary conditions for the case of a dyad only.

Referring to § 15, the general solution of the differential equation contains not only J functions, but also the functions $Y_2 = \frac{\cos y}{y} + \sin y$, which are suitable only for space not containing the origin. They are therefore suitable for any shell embracing an interior aggregate. In the shell the functions will be of the form AJ + BY, or as it may be written

$$\frac{\sin(\alpha + y)}{y} - \cos(\alpha + y).$$

It will be convenient to denote this by $f(\alpha, y)$.

Let now the radius of the interior aggregate be a, that of the exterior b. Let also λ, λ' denote the corresponding parameters.

Then we may write

$$\text{Inside} \quad \psi_1 = \mathrm{L}\left\{\mathrm{J}\left(\frac{\lambda r}{a}\right) - \frac{r^2}{a^2}\mathrm{J}\lambda\right\}\sin^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39),$$

$$\text{Shell} \quad \psi_2 = q\mathrm{L}\left\{f\left(\alpha, \frac{\lambda' r}{b}\right) - \frac{r^2}{b^2} f(\alpha, \lambda')\right\}\sin^2\theta \quad . \quad . \quad . \quad (40),$$

$$\text{Outside} \quad \psi = -\pi\mathrm{V}\left(r^2 - \frac{b^3}{r}\right)\sin^2\theta.$$

At the interface $\psi_1 = \psi_2$ and $\psi_1 = 0$, therefore

$$f\left(\alpha, \frac{\lambda' a}{b}\right) - \frac{a^2}{b^2} f(\alpha, \lambda') = 0.$$

Write $a/b \equiv p$. This equation, when developed, gives

$$\tan\alpha = -\frac{\mathrm{J}(\lambda' p) - p^2\mathrm{J}(\lambda')}{\mathrm{Y}(\lambda' p) - p^2\mathrm{Y}(\lambda')} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41).$$

Moreover, the tangential velocities must be the same. Hence, when $r = a$,

$$d\psi_1/dr = d\psi_2/dr.$$

Therefore

$$\frac{1}{a}\{\lambda \sin \lambda - 3\mathrm{J}(\lambda)\} = \frac{q}{bp}\{-f(\alpha, \lambda' p) + \lambda' p \sin(\alpha + \lambda' p) - 2p^2 f(\alpha, \lambda')\}.$$

But

$$f(\alpha, \lambda' p) = p^2 f(\alpha, \lambda').$$

Hence

$$\lambda \sin \lambda - 3\mathrm{J}(\lambda) = q\{\lambda' p \sin(\alpha + \lambda' p) - 3f(\alpha, \lambda' p)\} \quad . \quad . \quad . \quad (42).$$

So, also, making $d\psi_2/dr = d\psi/dr$ when $r = b$ we get

$$\mathrm{V} = \frac{q\mathrm{L}}{\pi b^2}\{f(\alpha\lambda') - \tfrac{1}{3}\lambda' \sin \lambda'\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43).$$

Equation (41) determines α; Equation (42) gives a relation between λ, λ', p, and q. We can therefore impress in general three further conditions. For instance, ratio of volumes, ratio of primary circulations, and ratio of secondary circulations.

There is a natural connection of the various singlets which go to make up an aggregate of the kind first discussed. At any interface all the differential co-efficients are continuous. In the polyad aggregates this is not so. Differential co-efficients beyond the first are not continuous. Monads, &c., which go to form them, are artificially combined. It is possible we may, on this basis, develop a theory of special aggregates which will unite with one another, or split up and be capable of uniting again in another manner. Some progress has been made with such a theory, but before an attempt is made to carry such a theory out it will be necessary to investigate the stability of the various systems. I hope soon to be able to take up this question.

[May 6, 1898.—By the permission of Professor HILL, to whose careful reading of the MS. I owe a great debt, I append an independent and very suggestive proof by him of the general theorem of gyrostatic vortices, based on the equations of motion.]

Take as co-ordinates r, θ, z.

$$\begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix}$$

Let p be the pressure,
ρ the density,
V the potential of the impressed forces.

Let τ be the velocity increasing r,
σ be the velocity increasing θ,
w be the velocity increasing z.

Then the equations of motions are

$$\left(\frac{d}{dt}+\tau\frac{d}{dr}+\sigma\frac{d}{r\,d\theta}+w\frac{d}{dz}\right)\tau-\frac{\sigma^2}{r}=-\frac{d}{dr}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

$$\left(\frac{d}{dt}+\tau\frac{d}{dr}+\sigma\frac{d}{r\,d\theta}+w\frac{d}{dz}\right)\sigma+\frac{\sigma\tau}{r}=-\frac{d}{r\,d\theta}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

$$\left(\frac{d}{dt}+\tau\frac{d}{dr}+\sigma\frac{d}{r\,d\theta}+w\frac{d}{dz}\right)w\qquad=-\frac{d}{dz}\left(\frac{p}{\rho}+\mathrm{V}\right)$$

$$\frac{d}{dr}(r\tau)+\frac{d\sigma}{d\theta}+\frac{d}{dz}(rw)=0.$$

It is desired to find a solution in which all the quantities are independent of θ.

Therefore

$$\frac{d\tau}{d\theta}=0,\quad\frac{d\sigma}{d\theta}=0,\quad\frac{dw}{d\theta}=0,\quad\frac{d}{d\theta}\left(\frac{p}{\rho}+\mathrm{V}\right)=0.$$

The last gives

$$\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)(r\sigma)=0.$$

If therefore ψ be the equation of a surface always containing the same particles of fluid, it is possible to take

$$r\sigma=\frac{1}{2\pi}f(\psi).$$

Also

$$\frac{d}{dr}(r\tau)+\frac{d}{dz}(rw)=0.$$

Let κ be the current function (which I distinguish throughout from ψ).

Therefore

$$\tau=-\frac{1}{2\pi r}\frac{d\kappa}{dz},\qquad w=\frac{1}{2\pi r}\frac{d\kappa}{dr}.$$

Substituting in

$$\left(\frac{d}{dt}+\tau\frac{d}{dr}+w\frac{d}{dz}\right)\psi=0,$$

it follows that

$$\frac{d\psi}{dt}-\frac{1}{2\pi r}\frac{d\kappa}{dz}\frac{d\psi}{dr}+\frac{1}{2\pi r}\frac{d\kappa}{dr}\frac{d\psi}{dz}=0.$$

Now make the further supposition that the surfaces $\psi=$ const. move without alteration parallel to the axis of z with velocity $\dot{\mathrm{Z}}$.

Therefore

$$\frac{d\psi}{dt}=-\dot{\mathrm{Z}}\frac{d\psi}{dz}.$$

Therefore

$$\frac{d\psi}{dr}\frac{d}{dz}(\kappa-\pi r^2\dot{\mathrm{Z}})-\frac{d\psi}{dz}\frac{d}{dr}(\kappa-\pi r^2\dot{\mathrm{Z}})=0.$$

Hence we can take

$$\kappa - \pi r^2 \dot{Z} = \psi.$$

Therefore

$$\tau = -\frac{1}{2\pi r}\frac{d\psi}{dz}, \qquad w = \frac{1}{2\pi r}\frac{d\psi}{dr} + \dot{Z}.$$

We have now

$$-\frac{d}{dr}\left(\frac{p}{\rho} + V + \frac{\tau^2 + w^2}{2}\right) = \frac{d\tau}{dt} + w\left(\frac{d\tau}{dz} - \frac{dw}{dr}\right) - \frac{\sigma^2}{r},$$

$$-\frac{d}{dz}\left(\frac{p}{\rho} + V + \frac{\tau^2 + w^2}{2}\right) = \frac{dw}{dt} - \tau\left(\frac{d\tau}{dz} - \frac{dw}{dr}\right).$$

Now

$$\frac{d\tau}{dt} = -\dot{Z}\frac{d\tau}{dz} = -\dot{Z}\frac{dw}{dr} - \dot{Z}\left(\frac{d\tau}{dz} - \frac{dw}{dr}\right),$$

$$\frac{dw}{dt} = -\dot{Z}\frac{dw}{dz}.$$

Therefore

$$-\frac{d}{dr}\left(\frac{p}{\rho} + V + \frac{\tau^2 + w^2}{2} - \dot{Z}w\right) = (w - \dot{Z})\left(\frac{d\tau}{dz} - \frac{dw}{dr}\right) - \frac{1}{4\pi^2 r^3}[f(\psi)]^2$$

$$= \frac{1}{2\pi r}\frac{d\psi}{dr}\left(\frac{d\tau}{dz} - \frac{dw}{dr}\right) - \frac{1}{4\pi^2 r^3}[f(\psi)]^2,$$

$$-\frac{d}{dz}\left(\frac{p}{\rho} + V + \frac{\tau^2 + w^2}{2} - \dot{Z}w\right) = -\tau\left(\frac{d\tau}{dz} - \frac{dw}{dr}\right)$$

$$= \frac{1}{2\pi r}\frac{d\psi}{dz}\left(\frac{d\tau}{dz} - \frac{dw}{dr}\right).$$

Hence

$$\frac{d}{dz}\left[\frac{d\psi}{dr}\left(\frac{\frac{d\tau}{dz} - \frac{dw}{dr}}{r}\right) - \frac{[f(\psi)]^2}{2\pi r^3}\right] = \frac{d}{dr}\left[\frac{d\psi}{dz}\left(\frac{\frac{d\tau}{dz} - \frac{dw}{dr}}{r}\right)\right].$$

Therefore

$$\frac{d\psi}{dr}\frac{d}{dz}\left(\frac{\frac{d\tau}{dz} - \frac{dw}{dr}}{r}\right) - \frac{d\psi}{dz}\left[\frac{d}{dr}\left(\frac{\frac{d\tau}{dz} - \frac{dw}{dr}}{r}\right) + \frac{f(\psi)f'(\psi)}{\pi r^3}\right] = 0.$$

Therefore

$$\frac{d\psi}{dr}\frac{d}{dz}\left[\frac{1}{r}\left(\frac{d\tau}{dz} - \frac{dw}{dr}\right) - \frac{f(\psi)f'(\psi)}{2\pi r^2}\right] - \frac{d\psi}{dz}\frac{d}{dr}\left[\frac{1}{r}\left(\frac{d\tau}{dz} - \frac{dw}{dr}\right) - \frac{f(\psi)f'(\psi)}{2\pi r^2}\right] = 0.$$

Therefore

$$\frac{1}{r}\left(\frac{d\tau}{dz} - \frac{dw}{dr}\right) = \frac{f(\psi)f'(\psi)}{2\pi r^2} + F'(\psi),$$

$$\frac{d^2\psi}{dr^2} - \frac{1}{r}\frac{d\psi}{dr} + \frac{d^2\psi}{dz^2} = -2\pi r^2 F'(\psi) - f(\psi)f'(\psi).$$

Therefore

$$-\frac{d}{dr}\left(\frac{p}{\rho}+\mathrm{V}+\frac{\tau^2+w^2}{2}-\dot{\mathrm{Z}}w\right)=\frac{1}{2\pi r}\frac{d\psi}{dr}\left[\frac{f(\psi)f'(\psi)}{2\pi r}+r\mathrm{F}'(\psi)\right]-\frac{[f(\psi)]^2}{4\pi^2r^3}$$

$$=\frac{d}{dr}\left[\frac{[f(\psi)]^2}{8\pi^2r^2}+\frac{\mathrm{F}(\psi)}{2\pi}\right],$$

and

$$-\frac{d}{dz}\left(\frac{p}{\rho}+\mathrm{V}+\frac{\tau^2+w^2}{2}-\dot{\mathrm{Z}}w\right)=\frac{1}{2\pi r}\frac{d\psi}{dz}\left[\frac{f(\psi)f'(\psi)}{2\pi r}+r\mathrm{F}'(\psi)\right]$$

$$=\frac{d}{dz}\left[\frac{[f(\psi)]^2}{8\pi^2r^2}+\frac{\mathrm{F}(\psi)}{2\pi}\right].$$

Therefore

$$\frac{p}{\rho}+\mathrm{V}+\frac{\tau^2+w^2}{2}-\dot{\mathrm{Z}}w+\frac{[f(\psi)]^2}{8\pi^2r^2}+\frac{\mathrm{F}(\psi)}{2\pi}=\text{arbitrary function of } t.$$

Therefore

$$\frac{p}{\rho}+\mathrm{V}+\tfrac{1}{2}[\tau^2+\sigma^2+(w-\dot{\mathrm{Z}})^2]+\frac{1}{2\pi}\mathrm{F}(\psi)=\text{arbitrary function of } t.$$

This arbitrary function of t is in this paper always a constant.

The last equation, together with the following, are the important equations :

$$\kappa=\psi+\pi\dot{\mathrm{Z}}r^2, \qquad \sigma=\frac{1}{2\pi r}f(\psi),$$

$$\tau=-\frac{1}{2\pi r}\frac{d\kappa}{dz}=-\frac{1}{2\pi r}\frac{d\psi}{dz},$$

$$w=\frac{1}{2\pi r}\frac{d\kappa}{dr}=\frac{1}{2\pi r}\frac{d\psi}{dr}+\dot{\mathrm{Z}},$$

$$\frac{d\tau}{dz}-\frac{dw}{dr}=\frac{1}{2\pi r}f(\psi)f'(\psi)+r\mathrm{F}'(\psi),$$

$$\frac{d^2\psi}{dr^2}-\frac{1}{r}\frac{d\psi}{dr}+\frac{d^2\psi}{dz^2}=-f(\psi)f'(\psi)-2\pi r^2\mathrm{F}'(\psi).$$

Whenever the conditions for the continuity of the τ and w components of the velocity have been satisfied at a separating surface whose equation is $\psi=$ const., then if the irrotational motion outside the surface have $\sigma=0$, we must have $\sigma=0$ when ψ is equal to the parameter of separating surface, if there is to be no slip there.

Therefore $f(\psi)=0$, when ψ is equal to the parameter of separating surface.

This is the case in the Third Section of the Paper.]

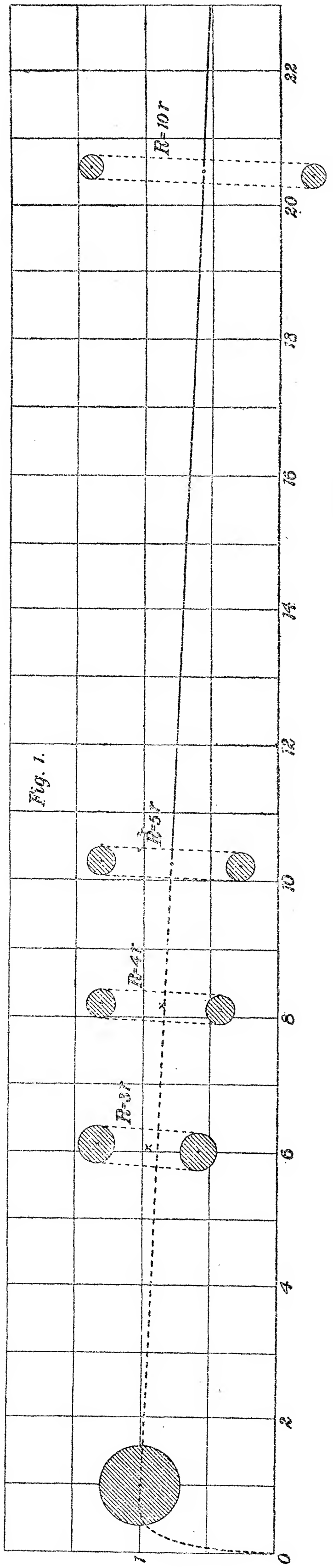
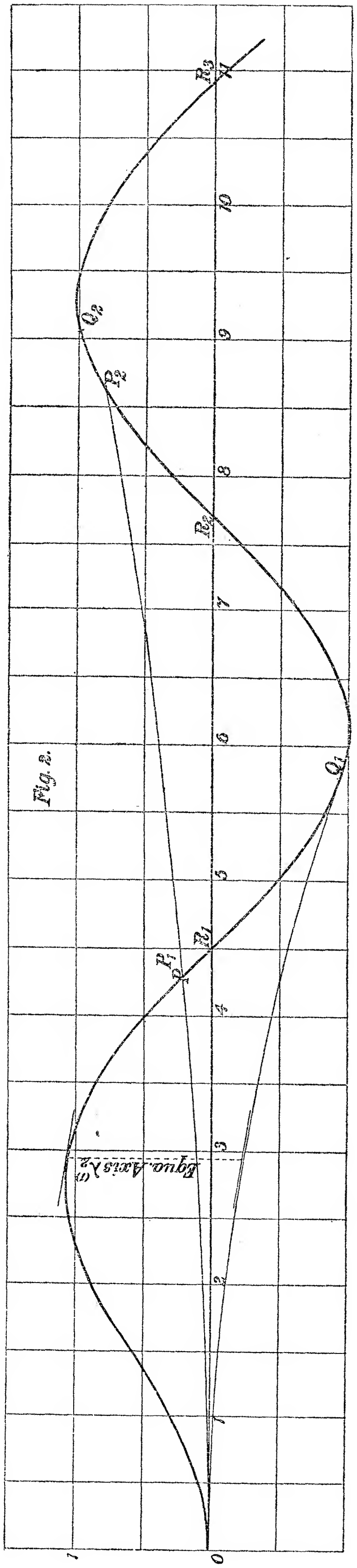


Fig. 1.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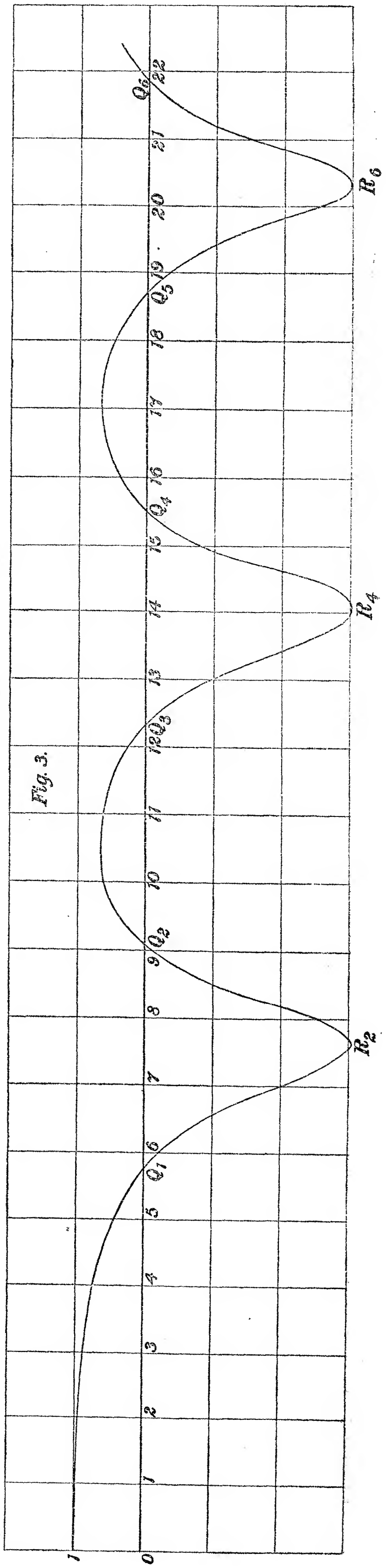
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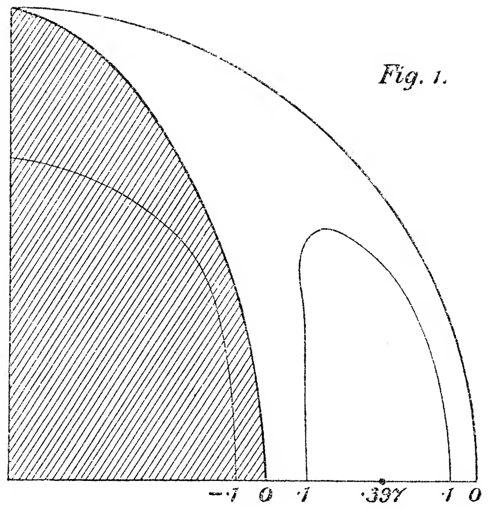
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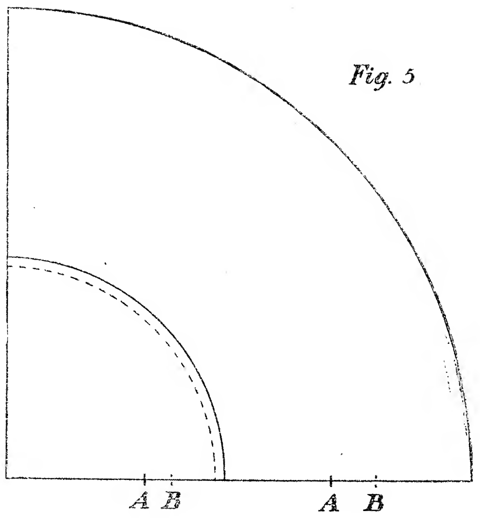
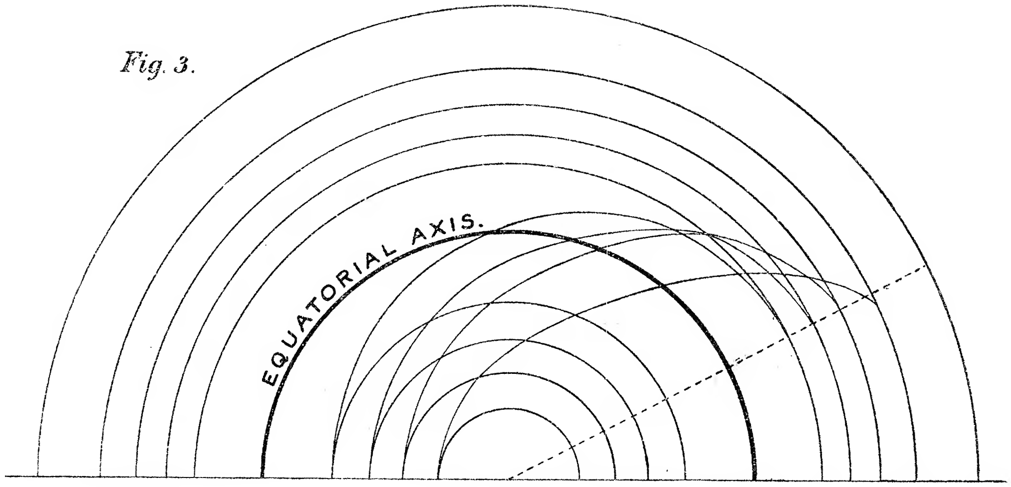


Fig. 5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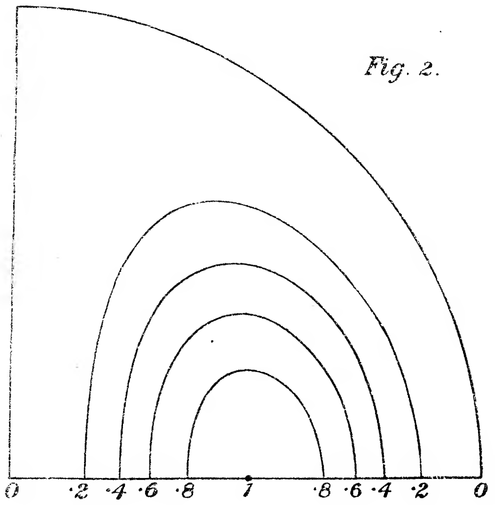
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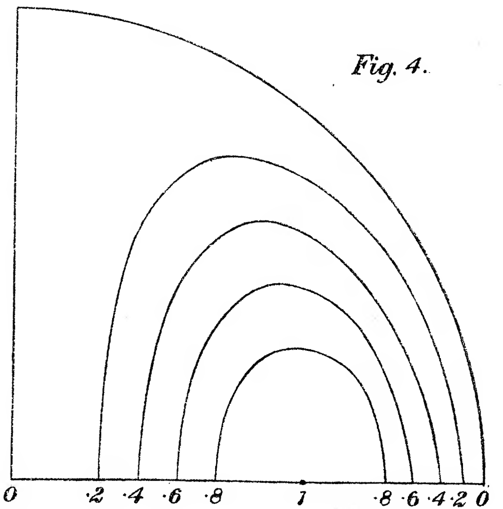
Fig. 4.